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Confinement with Kalb-Ramond fields

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ABSTRACT: We consider models with N U(1) gauge fields A_μ^n, N Kalb-Ramond fields $B_{\mu\nu}^n$, an arbitrary bare action and a fixed UV cutoff Λ. Under mild assumptions these can be obtained as effective low energy theories of SU$(N+1)$ Yang Mills theories in the maximal abelian gauge. For a large class of bare actions they can be solved in the large-N limit and exhibit confinement. The confining phase is characterized by an approximate "low energy" vector gauge symmetry under which the Kalb-Ramond fields $B_{\mu\nu}^n$ transform. The same symmetry allows for a duality transformation showing that magnetic monopoles have condensed. The models allow for various mechanisms of confinement, depending on which sources for A_μ^n or $B_{\mu\nu}^n$ are switched on, but the area law for the Wilson loop is obtained in any case.

KEYWORDS: 1/N Expansion, Confinement, Duality in Gauge Field Theories.

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1. Introduction

Monopole condensation is widely believed to be the mechanism responsible for confinement in Yang-Mills (YM) theories [1, 2] (see [3] for a review). The picture of a confining vacuum as a dual superconductor gives rise to models for the low energy behaviour of YM theories which are based on (the dual of) the abelian Higgs model [4]. Albeit quite successful phenomenologically [5, 6], it is practically impossible to derive these models in a systematic way from YM theories: by construction (i.e. the assumption of duality) the gauge and scalar fields of the dual models are related non-locally to the YM gauge fields, at least off-shell, hence local dual models can only capture the semi-classical features of low energy YM theories.

Generalizing a formalism proposed by Julia and Toulouse [7] Quevedo and Trugenberger [8] have presented quadratic actions for antisymmetric tensor fields, which describe the condensation of various topological defects in various space-time dimensions d. If applied to monopole condensation in $d = 4$, this approach suggests a (local) action for a Kalb-Ramond field $B_{\mu\nu}$ [9].

There are two ways to understand the usefulness of 2-form fields $B_{\mu\nu}$ for the description of monopole condensation in $d = 4$: first, the dual abelian Higgs model in the broken phase necessarily has to contain a pseudoscalar Goldstone boson. Reversing the duality transformation in $d = 4$, the dual of a pseudoscalar is a 2-form field $B_{\mu\nu}$. Second, in the presence of a monopole the abelian Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0$

is violated on the world-line $x(t)$, and the field strength $F_{\mu\nu}$ cannot be written in the form $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ in $x(t)$. If monopoles have condensed in the vacuum, the abelian Bianchi identity is violated everywhere, and the field $F_{\mu\nu}$ is nowhere of the form $\partial_{[\mu}A_{\nu]}$. In this situation it is useful to introduce an auxiliary field $B_{\mu\nu}$ for $F_{\mu\nu}$ [10] in some analogy to the field strength formulation of YM theories [11] (it suffices, however, to introduce auxiliary fields $B_{\mu\nu}$ for the components of $F_{\mu\nu}$ associated to the $N_c - 1$ U(1) subgroups of SU(N_c), see below).

All this does not imply that a 2-form field $B_{\mu\nu}$ appears in the physical spectrum of a theory with condensed monopoles: the equations of motion for $B_{\mu\nu}$, as derived from the full effective action, can well be algebraic, i.e. the propagator of $B_{\mu\nu}$ has not necessarily a simple pole for finite negative p^2 (in the minkowskian regime). The role of the $B_{\mu\nu}$ dependent terms in the full effective action is then just to parametrize in a compact way a subclass of higher derivative interactions among the fundamental gauge fields A_μ, but the modes in $B_{\mu\nu}$ which are not of the form $\partial_{[\mu}A_{\nu]}$ correspond to topologically non-trivial gauge field configurations A_μ associated to a non-vanishing monopole density.

Monopole condensation is a purely abelian phenomenon even in the context of YM theories. Abelian dominance becomes particularly clear [12] in the maximal abelian gauge (MAG) [13]: the $N_c^2 - 1$ gauge fields of SU(N_c) are divided into the $N = N_c - 1$ U(1) gauge fields A_μ^n, $n = 1, \ldots, N$, associated to the generators of the U$(1)^N$ Heisenberg subalgebra, and $N_c^2 - N_c$ "charged" gauge fields W_μ^a (which carry at least one of the N U(1) charges) associated to the off-diagonal generators. The continuum MAG is characterized by the gauge condition $(D_\mu W_\mu)^a = 0$, where D_μ are the U$(1)^N$-covariant derivatives. Lattice results indicate that the W_μ^a gauge fields are massive in the MAG, with $M_W \sim 1\,\mathrm{GeV}$ (for SU(2)) [14]. A mechanism for W_μ mass generation based on ghost condensates has been proposed in [15].

Combining the above considerations one is led to conclude that the natural low energy degrees of freedom of confining gauge theories are N U(1) gauge fields A_μ^n, and N Kalb-Ramond fields $B_{\mu\nu}^n$ which serve to describe monopole condensation. (Here "degree of freedom" does *not* necessarily signify an asymptotic physical state, but possibly just an auxiliary field with algebraic equations of motion.)

The purpose of the present paper is the solution of a field theory with the corresponding field content, with a bare action as general as possible, in the large-N limit. The idea is to treat this theory as an effective low energy theory valid below some scale $\Lambda \sim M_W$ (with, e.g., $\Lambda \sim 1\,\mathrm{GeV}$). Consequently we will supplement the model with an UV cutoff Λ, and allow for arbitrary non-renormalizable interactions $\sim \Lambda^{-p}$ (where p is determined by power counting) in the bare action. The precise numerical value of Λ will not concern us here.

As we will find below, these models indeed describe confinement provided the parameters of the bare action satisfy some inequality. In the confining phase a duality transformation to an abelian Higgs model can be specified (although the full dual action is still non-local), which shows that one describes indeed monopole condensation. Remarkably the effective action exhibits an approximate vector-like "gauge symmetry" (a slight exaggeration, since it is broken by derivative terms) which allows to "gauge away" the low momentum modes of the U(1) gauge fields A_μ^n. They are swallowed by the fields $B_{\mu\nu}^n$ in analogy with the Higgs-Kibble phenomenon with an additional Lorentz index. This symmetry of the low energy effective action coincides with the one in [9] for the quadratic terms of the action. Here, however, it holds for part of the interactions (to arbitrary powers in the fields) as well and, simultaneously, renders a more general duality transformation possible.

The conventional criterium for confinement is the area law of the expectation value of the Wilson loop. In an abelian theory the Stokes theorem allows to express the Wilson loop in terms of a source localized on a "Wilson surface" bounded by the Wilson loop. In our model such a source can be coupled either to $F_{\mu\nu}$ (denoted by J^F), to $B_{\mu\nu}$ (denoted by J^B) or both of them. As we will discuss in detail, in the confining phase of the model the area law is obtained in any case. We will show, on the other hand, that only a particular linear combination of the sources J^F and J^B is consistent, once the model is coupled to external quantum fields: fluctuations of such quantum fields would generate an infinite contribution to the action unless the long-range parts of the correlators are cancelled. The remaining short-range parts of the correlators coincide with various models for the expectation value of Yang-Mills field strength correlators [6, 16].

A full-fledged computation of the string tension would require the solution of a coupled set of equations of motion (in analogy to approaches based on the dual abelian Higgs model [6]) which will not be attempted here. In any case the result will be "non-universal" in the sense that it depends on details of the bare action and the UV cutoff.

In this paper we focus rather on the infrared regime of a confining gauge theory and the remarkable fact that, given the above field content representing the feature of "abelian dominance" and a non-perturbative computational scheme like the $1/N$ expansion, such theories can actually be solved in spite of confinement.

The subsequent sections of the paper are organized as follows: in section 2 we will discuss more precisely, how the model can be obtained from SU(N_c) YM theories, specify its field content and parametrize its bare action. Section 3 is devoted to its solution at large N. Since the visibility of the confining phase necessitates a careful treatment of the limit of an infrared cutoff k^2 tending to zero, we will first review the bosonic O(N) model at large N in the broken phase: already in this simple model a convex effective potential, and hence a consistent treatment of the broken phase, requires the introduction of $k^2 \neq 0$ and a discussion of non-analyticities for $k^2 \to 0$. Subsequently the emergence of a confining phase of the A_μ-$B_{\mu\nu}$–model is seen to share similar non-analyticities with the O(N) model, which originate again from the convexity of the effective action (and *not* from the large-N limit).

In section 4 we will discuss various properties of the confining phase: the possibility of a duality transformation, the emergence of a vector-like approximate gauge symmetry, the effective propagators of the $A_\mu - B_{\mu\nu}$ – system, the "response" of the system to external sources like the Wilson loop and the unsuccessful search for bound states (hence "glueballs" will only emerge once the charged gluons of the SU(N_c) YM theory is taken into account). In section 5 we compare the large-N limit of our models to the large-N_c limit of SU(N_c) YM (which turns out to be not the same), discuss generalizations of the bare action considered in section 3 as well as the notion of universality, and conclude with an outlook. A short summary of the subsequent results has already appeared in [25].

2. The A_μ-$B_{\mu\nu}$–model

Before giving a detailed description of our model, we will motivate it starting with the pure SU(N_c) YM theory in the MAG. As noted in the introduction, one splits the gauge fields into the $N = N_c - 1$ U(1) gauge fields A^n_μ and the charged gauge fields W^a_μ. The euclidean partition functions reads

$$\frac{1}{\mathcal{N}'}\int \mathcal{D}A\ \mathcal{D}W\mathcal{D}c\mathcal{D}\bar{c}\, e^{-S_{\mathrm{YM}}-S_{gf}-S_{gh}} \tag{2.1}$$

where the YM action can be decomposed as

$$S_{\mathrm{YM}} = \int d^4x\,\{F^n_{\mu\nu}F^n_{\mu\nu} + F^a_{\mu\nu}F^a_{\mu\nu}\} \tag{2.2}$$

and $\mathcal{N}'$ is a normalization.

The gauge fixing terms are of the form

$$S_{gf} = \int d^4x \left\{\frac{1}{2\alpha}\left(\partial_\mu A^n_\mu\right)^2 + \frac{1}{2\alpha_W}\left(D_\mu W^a_\mu\right)^2\right\} \tag{2.3}$$

with the U(1)^N-covariant derivative

$$D_\mu W^a_\mu = \partial_\mu W^a_\mu + g\, f^a_{\ nb}\, A^n_\mu\ W^b_\mu\,. \tag{2.4}$$

The form of the ghost interactions in S_{gh} does not play any role subsequently.

Now we introduce Kalb-Ramond fields $B^n_{\mu\nu}$ as collective fields for the N field strengths $F^n_{\mu\nu}$ (*including* their non-abelian terms $\sim g\ f^n_{\ ab}\ W^a_{[\mu}W^b_{\nu]}$) in the partition function (2.1). This amounts to the multiplication of the integrand of (2.1) by

$$1 = \frac{1}{\mathcal{N}_B}\int \mathcal{D}B\ e^{-S_B}\,, \qquad S_B = \int d^4x \left\{\frac{1}{4}\left(F^n_{\mu\nu} - B^n_{\mu\nu}\right)^2\right\}. \tag{2.5}$$

The new version of the partition function reads

$$\frac{1}{\mathcal{N}}\int \mathcal{D}A\mathcal{D}B\mathcal{D}W\mathcal{D}c\mathcal{D}\bar{c}\, e^{-S_{\mathrm{YM}}-S_{gf}-S_{gh}-S_B} \tag{2.6}$$

with $\mathcal{N} = \mathcal{N}'\mathcal{N}_B$.

Next, as stated in the introduction, we assume that the $\mathcal{D}W$ path integral is "infrared save": lattice results show that the W_μ^a gauge fields in the MAG aquire a finite mass M_W, in agreement with the phenomenon of infrared abelian dominance in this gauge [14]. A mechanism for W_μ^a mass generation which involves the four ghost interactions in S_{gh} in the MAG has been proposed in [15]. The same mechanism (bi-ghost condensation) renders the "charged" ghosts c^a, $\bar{c}^a$ massive. Now we assume that the $\mathcal{D}W$ and $\mathcal{D}c\mathcal{D}\bar{c}$ path integrals in (2.6) are performed. However, in order not to generate uncontrollable UV divergences, it is advisable to integrate simultaneously over the "high momentum" modes of A_μ^n and $B_{\mu\nu}^n$: these modes are needed in order to cancel UV divergences which would violate the Slavnov-Taylor identities.

Let us thus split the measures $\mathcal{D}A$ and $\mathcal{D}B$ as

$$\mathcal{D}A = \mathcal{D}A_{<M_W} \cdot \mathcal{D}A_{>M_W}\ , \qquad \mathcal{D}B = \mathcal{D}B_{<M_W} \cdot \mathcal{D}B_{>M_W}\ . \tag{2.7}$$

The indices $<M_W$ and $>M_W$ denote modes with momenta in the corresponding ranges. Now we rewrite the partition function (2.6) as

$$\frac{1}{\mathcal{N}_<} \int \mathcal{D}A_{<M_W}\, \mathcal{D}B_{<M_W}\, e^{-S_{\text{bare}}(A,B,M_W) - S_{gf}} \tag{2.8}$$

with

$$e^{-S_{\text{bare}}(A,B,M_W) - S_{gf}} = \frac{\mathcal{N}_<}{\mathcal{N}} \int \mathcal{D}A_{>M_W}\, \mathcal{D}B_{>M_W}\ \mathcal{D}W\ \mathcal{D}c\mathcal{D}\bar{c}\, e^{-S_{\text{YM}} - S_{gf} - S_{gh} - S_b}\ . \tag{2.9}$$

S_{gf} in (2.8) and on the left hand side of (2.9) corresponds to the first (abelian) term in (2.3). The partition function (2.8) corresponds to the class of models which are the subject of the present paper. It represents the "effective low energy theory" of SU(N_c) YM under the only assumption that $S_{\text{bare}}(A, B, M_W)$ is local in the sense that an expansion in positive (but otherwise arbitrary) powers of derivatives exists, and that the terms of lowest order (without derivatives on $B_{\mu\nu}$ and $F_{\mu\nu}$) are non-zero. Its dependence on M_W is then dictated by power counting.

Since the split of the path integral measures in (2.7)–(2.9) is somewhat formal, we will very briefly present an alternative definition of the model. To this end we introduce in the path integral (2.6) a) sources $J_{A,\mu}^n$ and $J_{B,\mu\nu}^n$ for A_μ^n and $B_{\mu\nu}^n$, and b) an infrared cutoff for A_μ^n and $B_{\mu\nu}^n$ in the form of a quadratic term $(-\Delta S_k(A, B))$ in the exponent, with

$$\Delta S_k(A,B) = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left\{ A_\mu^n(p)\, R_{\mu\nu}^A(k,p)\, A_\nu^n(-p) + B_{\mu\nu}^n(p)\, R_{\mu\nu,\rho\sigma}^B(k,p)\, B_{\rho\sigma}^n(-p) \right\} . \tag{2.10}$$

The tensors R^A and R^B should suppress low momentum modes, i.e. become large or even diverge for $p^2 < k^2$ (without having zero modes). First we choose $k = M_W$ and define

$$e^{-W_{M_W}(J_A,J_B)} = \frac{1}{\mathcal{N}} \int \mathcal{D}A\ \mathcal{D}B\ \mathcal{D}W\ \mathcal{D}c\mathcal{D}\bar{c}\ e^{-S_{\text{YM}} - \Delta S_{M_W} - S_{gf} - S_{gh} - S_B + J_A \cdot A + J_B \cdot B}\ . \tag{2.11}$$

The Legendre transform of $W_{M_W}(J_A, J_B)$ with respect to J_A, J_B, the effective action $\Gamma_{M_W}(A, B)$, plays the same role as $S_{\text{bare}}(A, B, M_W)$ in (2.8).

It is staightforward to replace M_W in (2.11) by a varying IR cutoff k, take the derivative of both sides of eq. (2.11) with respect to k, and derive an exact wilsonian renormalization equation [17] for $W_k(J_A, J_B)$ or its Legendre transform $\Gamma_k(A, B)$. The A_μ-$B_{\mu\nu}$–model is then *defined* by the corresponding wilsonian RGE for W_k or Γ_k, and the boundary condition at $k = M_W$ given by (2.11). (The advantage of this procedure is that Slavnov-Taylor identities of the underlying SU(N_c) YM theory are well under control [18] even in the presence of cutoffs as in (2.10)). For convenience we will, however, most of the time refer to (2.8) as the definition of the model.

Apart by locality, $S_{\text{bare}}(A, B, M_W)$ in (2.8) is constrained by the $N = N_c - 1$ U(1) gauge symmetries preserved by the second term in S_{gf} in (2.3), up to a covariant linear gauge fixing term corresponding to the first term in (2.3) (this can easily be proven by deriving a corresponding Ward identity). In the absence of fields carrying U(1) charges S_{bare} can thus depend on A^n_μ only through its abelian field strengths $F^n_{\mu\nu}$, hence we write $S_{\text{bare}}(F, B, M_W)$. However, S_{bare} is not constrained by renormalizability; we have to allow for arbitrary powers in the fields and momenta. Subsequently we assume that all dimensionful parameters are proportional to corresponding powers of M_W (which also plays the role of an UV cutoff $\Lambda = M_W$, cf. (2.8)) and omit the corresponding powers of M_W.

Still, the most general dependence of S_{bare} on F and B does not yet allow to solve the model in a large-N limit. To this end one has to assume that S_{bare} depends only on O(N) invariant bilinear operators (quadratic in F or B). This corresponds to the assumption that the low energy effective action of SU(N_c) YM is dominated by quadratic SU(N_c) invariants; then O(N), under which $F^n_{\mu\nu}$ and $B^n_{\mu\nu}$ transform as N-plets, appears as an accidental global symmetry. As we will show later the qualitative features of our results do, however, not depend on the large-N limit and hence not on the global O(N) symmetry.

Let us define the following three Lorentz scalar bilinear operators

$$\begin{aligned}
\mathcal{O}_1(x) &= \sum_{n=1}^{N} F^n_{\mu\nu}(x)\, F^n_{\mu\nu}(x)\,, \\
\mathcal{O}_2(x) &= \sum_{n=1}^{N} F^n_{\mu\nu}(x)\ B^n_{\mu\nu}(x)\,, \\
\mathcal{O}_3(x) &= \sum_{n=1}^{N} B^n_{\mu\nu}(x)\ B^n_{\mu\nu}(x)\,. \qquad (2.12)
\end{aligned}$$

One of the simplest bare actions of the present class of models, which exhibits already all of the essential features, is given by

$$S_{\text{bare}}(F, B) = \int d^4x \left\{ \mathcal{L}_{\text{bare}}(\mathcal{O}_i) + \frac{h}{2}\left(\partial_\mu \widetilde{B}^n_{\mu\nu}\right)^2 + \frac{\sigma}{2}\left(\partial_\mu B^n_{\mu\nu}\right)^2 \right\} \qquad (2.13)$$

with

$$\widetilde{B}^n_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\ B^n_{\rho\sigma}\,. \tag{2.14}$$

$\mathcal{L}_{\rm bare}$ in (2.13) may contain derivatives acting on the operators $\mathcal{O}_i$ defined in (2.12). Somewhat ad hoc we choose just bilinear "kinetic" terms for $B^n_{\mu\nu}$; generalizations are straightforward and will be discussed in section 5. An artefact of the ansatz (2.13) with constant h, σ will be the appearance of asymptotic states (poles in the propagators) with masses $\sim M_W^2/h$; we do not expect the corresponding states to appear in an effective low energy theory of SU(N_c) YM, since these would be N-plets of O(N). They could easily be avoided by replacing h, σ in (2.13) by "higher derivative" terms

$$h(p^2)\,,\ \sigma(p^2) \quad \text{with} \quad h(p^2)\,,\ \sigma(p^2) \to 0 \quad \text{for} \quad |p^2| \gtrsim M_W^2 \tag{2.15}$$

as is to be expected once these terms are generated by loops of the "charged" sector of SU(N_c) YM. Our subsequent main results will not depend on this choice, and unless stated otherwise we will continue to work with constant h, σ for simplicity.

In order to solve the model in the large-N limit we have to make assumptions on the N dependence of the parameters in $S_{\rm bare}$. These assumptions can be summarized by rewriting (2.13) as

$$S_{\rm bare}(F,B) = \int d^4x \left\{ N\mathcal{L}'_{\rm bare}\left(\frac{\mathcal{O}_i}{N}\right) + \frac{h}{2}\left(\partial_\mu \widetilde{B}^n_{\mu\nu}\right)^2 + \frac{\sigma}{2}\left(\partial_\mu B^n_{\mu\nu}\right)^2 \right\}, \tag{2.16}$$

where the coefficients of $\mathcal{L}'_{\rm bare}$ are independent of N.

Our subsequent aim is the computation of the (euclidean) generating functional $W(J)$ of connected Green functions,

$$e^{-W(J)} = \frac{1}{\mathcal{N}} \int \mathcal{D}A_{<M_W}\,\mathcal{D}B_{<M_W}\ e^{-S_{\rm bare}(F,B) + \int d^4x \left\{\frac{1}{2\alpha}(\partial_\mu A^n_\mu)^2 + J^n_{A,\mu}A^n_\mu + J^n_{B,\mu\nu}B^n_{\mu\nu}\right\}} \tag{2.17}$$

and the effective action obtained by the Legendre transform of $W(J)$. The indices $< M_W$ denote modes with momenta $p^2 \lesssim M_W^2$ as discussed below eq. (2.7). To this end we can add cutoff terms $\Delta S(A,B)$ to the exponent in (2.17), which have been defined in (2.10). Now, however, the cutoff functions R^A and R^B should suppress high momentum modes with $p^2 \gtrsim M_W^2$.

3. The large-N solution

The most convenient formalism for the solution of models of the form (2.16) in the large-N limit is the introduction of auxiliary O(N) singlet fields [19, 20], which we will use as well. Already in the case of bosonic O(N) models with spontaneously broken symmetry there are, however, some subtleties associated to the convexity of the effective action: in order to "see" the convexity of the effective potential in the broken phase it is necessary to introduce an artificial infrared cutoff k^2 (in, e.g., momentum space) and to study carefully the limit $k^2 \to 0$. Conventionally this phenomenon is investigated in the context of the wilsonian Exact Renormalization Group [21], but this formalism is not obligatory.

In the case of models of the form (2.16) we observe a similar phenomenon: now the "confining phase" is observed only in the limit of an infrared cutoff k^2 tending to zero, and again its existence is related to the convexity of the effective action.

In order to review these points, which play an essential role in our class of models, we will first discuss them in the context of the bosonic O(N) model in the broken phase in the next section 3.1. In the section 3.2 we will return to the A_μ-$B_{\mu\nu}$–model and use what we have learned before.

3.1 The solution of the scalar O(N)-model

The field content of this model is given by an N-plet of scalars φ^n , $n = 1, \ldots, N$. It is convenient to introduce a composite O(N) singlet operator

$$\mathcal{O}(x) = \varphi^n(x)\varphi^n(x)\,. \tag{3.1}$$

The bare action of the model contains a bare potential, which can conveniently be expressed in terms of the operator $\mathcal{O}(x)$. If one insists on renormalizability the bare potential contains only terms linear and quadratic in $\mathcal{O}(x)$. Including its generic N dependence it is then of the form

$$V_{\text{bare}}\left(\frac{\mathcal{O}}{N}\right) = \frac{1}{N}\int d^4x \left\{-\frac{m^2}{2}\,\mathcal{O}(x) + \frac{\lambda}{4N}\,\mathcal{O}(x)^2\right\}\,. \tag{3.2}$$

(We chose a minus sign in front of m^2 in order to have m^2 positive in the broken phase.) In order to implement both an infrared cutoff k and an UV cutoff Λ we supplement the kinetic terms with a cutoff term ΔS_k^Λ:

$$S_{kin}(\varphi) = \frac{1}{2}\int d^4x\,\partial_\mu\varphi^n\partial_\mu\varphi^n + \Delta S_k^\Lambda\,, \tag{3.3}$$

where

$$\Delta S_k^\Lambda = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\varphi(p)\,R_k^\Lambda(p^2)\,\varphi(-p)\,. \tag{3.4}$$

The cutoff function R_k^Λ diverges for $p^2 \lesssim k^2$ and $p^2 \gtrsim \Lambda^2$, and vanishes for $k^2 \lesssim p^2 \lesssim \Lambda^2$. The aim is to compute the generating functional $W_k(J)$ in the limit $k \to 0$, where

$$e^{-W_k(J)} = \frac{1}{\mathcal{N}'}\int \mathcal{D}\varphi\, e^{-S_{kin}(\varphi) - NV_{\text{bare}}\left(\frac{\mathcal{O}}{N}\right) + \int d^4x\, J^n\varphi^n}\,. \tag{3.5}$$

The UV cutoff Λ is considered to be fixed, and $\mathcal{N}'$ ensures $W_0(0) = 0$. The auxiliary field method consists in writing

$$e^{-NV_{\text{bare}}\left(\frac{\mathcal{O}}{N}\right)} = \frac{1}{\mathcal{N}_\phi}\int \mathcal{D}\phi\, e^{-NG_{\text{bare}}(\phi) - \int d^4x\,\phi\cdot\mathcal{O}}\,. \tag{3.6}$$

In the large-N limit the path integral in (3.6) is dominated by the stationary point(s), and the relation between $V_{\rm bare}$ and $G_{\rm bare}$ becomes a Legendre transformation:

$$N V_{\rm bare}\left(\frac{\mathcal{O}}{N}\right) = N G_{\rm bare}(\widehat{\phi}) + \int d^4x\, \widehat{\phi}\cdot\mathcal{O} \tag{3.7}$$

where $\widehat{\phi} \equiv \widehat{\phi}(\mathcal{O})$ solves

$$\left[\frac{\delta}{\delta\phi(x)} N G_{\rm bare}(\phi)\right]_{\widehat{\phi}(\mathcal{O})} = -\,\mathcal{O}(x)\,. \tag{3.8}$$

Conversely we have

$$\left[\frac{\delta}{\delta\mathcal{O}(x)} N V_{\rm bare}\left(\frac{\mathcal{O}}{N}\right)\right]_{\mathcal{O}(\widehat{\phi})} = \widehat{\phi}(x)\,. \tag{3.9}$$

The correctness of the assignment of the powers of N in (3.7) can be verified by rescaling $\mathcal{O} \to N\mathcal{O}'$, after which each term is proportional to N and the factor N can be dropped.

Generally, for given $V_{\rm bare}$, one can solve (3.9) for $\mathcal{O}(\widehat{\phi})$, insert it into (3.7) and obtain $G_{\rm bare}(\widehat{\phi})$. With $V_{\rm bare}$ as in (3.2) one obtains

$$G_{\rm bare}(\phi) = -\frac{1}{\lambda}\int d^4x \left(\phi + \frac{m^2}{2}\right)^2, \tag{3.10}$$

$$\widehat{\phi}(\mathcal{O}) = -\frac{m^2}{2} + \frac{\lambda}{2N}\,\mathcal{O}(x)\,. \tag{3.11}$$

Note that $G_{\rm bare}(\phi_i)$ is negative and unbounded from below for $\phi_i \to \infty$, which is no point of concern as long as ϕ is an auxiliary field with algebraic equations of motion. Using (3.6) the expression (3.5) can be written as

$$e^{-W_k(J)} = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi\, e^{-NG_{\rm bare}(\phi)} \int \mathcal{D}\varphi\, e^{-S_{kin}(\varphi) - \int d^4x(\phi\cdot\mathcal{O} - J^n\varphi^n)} \tag{3.12}$$

with $\mathcal{N} = \mathcal{N}'\mathcal{N}_\phi$.

The $\mathcal{D}\varphi$ path integral is thus gaussian and gives

$$e^{-W_k(J)} = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi\, e^{-NG_{\rm bare}(\phi) - N\Delta G(\phi) + \frac{1}{2}\int d^4x_1 d^4x_2 J^n(x_1) P(x_1,x_2,\phi) J^n(x_2)} \tag{3.13}$$

with

$$\Delta G(\phi) = -\frac{1}{2}\,{\rm Tr}\log\left(P(x_1,x_2,\phi)\right)\,. \tag{3.14}$$

(A factor N from the trace in (3.14) has already been included in (3.13).) The propagator $P(x_1,x_2,\phi)$ has to be obtained from the quadratic terms in φ in the exponent in (3.12); it satisfies

$$\int d^4x_2 \left[\frac{\delta^2 S_{kin}(\varphi)}{\delta\varphi^n(x_1)\,\delta\varphi^m(x_2)} + 2\delta_{nm}\,\delta^4(x_1-x_2)\phi(x_1)\right] P^{mk}(x_2,x_3,\phi) = \delta_{nk}\,\delta^4(x_1-x_3) \tag{3.15}$$

and we always use $P^{mk} \equiv \delta_{mk}P$.

For constant ϕ its Fourier transform reads

$$\widetilde{P}(p^2, \phi) = \frac{1}{p^2 + R_k^\Lambda(p^2) + 2\phi} \tag{3.16}$$

and (3.14) becomes

$$\Delta G(\phi) = \frac{1}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \log\left(p^2 + R_k^\Lambda(p^2) + 2\phi\right). \tag{3.17}$$

For what follows it is most convenient to employ a "sharp" cutoff function $R_k^\Lambda(p^2)$, which is infinite for $p^2 < k^2$ and $p^2 > \Lambda^2$ and zero for $k^2 < p^2 < \Lambda^2$. Up to irrelevant terms independent of ϕ this gives the same result as corresponding cutoffs of the d^4p integral, and one obtains

$$\begin{aligned} \Delta G(\phi) = \frac{1}{64\pi^2} \int d^4x \Big\{ &\left(\Lambda^4 - 4\phi^2\right) \log\left(\Lambda^2 + 2\phi\right) - \\ &- \left(k^2 - 4\phi^2\right) \log\left(k^2 + 2\phi\right) + 2\phi\left(\Lambda^2 - k^2\right) \Big\}. \end{aligned} \tag{3.18}$$

Again, in the large-N limit the path integral (3.13) can be replaced by the stationary point, and $W_k(J)$ becomes

$$W_k(J) = N G_{\text{bare}}(\widehat{\phi}) + N \Delta G(\widehat{\phi}) - \frac{1}{2} \int d^4x_1 d^4x_2 \left\{ J^n(x_1) P(x_1, x_2, \widehat{\phi}) J^n(x_2) \right\}, \tag{3.19}$$

where $\widehat{\phi} \equiv \widehat{\phi}(J)$ is the configuration which extremizes the right-hand side of (3.19). First we look for the vacuum configuration $\widehat{\phi}_0 \equiv \widehat{\phi}(J = 0)$ which has to solve, with G_{bare} from (3.10) and ΔG from (3.18),

$$\begin{aligned} 0 &= \left[\frac{\delta}{\delta\phi} \left(G_{\text{bare}}(\phi) + \Delta G(\phi)\right) \right]_{\widehat{\phi}_0} \\ &= \left[-\frac{1}{\lambda}(2\phi + m^2) + \frac{1}{16\pi^2} \left(2\phi \log\left(\frac{k^2 + 2\phi}{\Lambda^2 + 2\phi} \right) + \Lambda^2 - k^2 \right) \right]_{\widehat{\phi}_0}. \end{aligned} \tag{3.20}$$

Clearly the solution $\widehat{\phi}_0$ of (3.20) depends on the infra-red cutoff k^2. It is instructive to follow the solution $\widehat{\phi}_0$, as a function of the infrared cutoff k^2, from $k^2 = \Lambda^2$ down to $k^2 \to 0$. For $k^2 = \Lambda^2$, where $\Delta G(\phi)$ vanishes, we have

$$\widehat{\phi}_0 \Big|_{k^2 = \Lambda^2} = -\frac{m^2}{2} \tag{3.21}$$

which is negative by assumption and our convention in (3.2). For $k^2 < \Lambda^2$ the solution $\widehat{\phi}_0$ has always to correspond to a positive argument of the logarithm in (3.20), which implies (assuming $\Lambda^2 > m^2$, i.e. $\Lambda^2 + 2\widehat{\phi}_0 > 0$)

$$\widehat{\phi}_0 \Big|_{k^2} > -\frac{k^2}{2}. \tag{3.22}$$

Under the additional assumption

$$\frac{\Lambda^2}{16\pi^2} < \frac{m^2}{\lambda} \tag{3.23}$$

the following subtle behaviour is observed in the limit of vanishing infrared cutoff k^2: one has simultaneously

$$\begin{aligned} k^2 &\to 0\,, \\ \widehat{\phi}_0 &\to 0_{-\varepsilon}\,, \\ \frac{1}{8\pi^2}\,\widehat{\phi}_0 \log\left(\frac{k^2+2\widehat{\phi}_0}{\Lambda^2}\right) &\to \frac{m^2}{\lambda} - \frac{\Lambda^2}{16\pi^2} = \text{finite}\,. \end{aligned} \tag{3.24}$$

Note that the non-analytic behaviour (3.24) could not be obtained if one would solve (3.20) right away for $k^2 = 0$. It corresponds in fact to the flat "inner region" of the effective potential $V_{\text{eff}}(\varphi^n)$ in the broken phase, which is required by the convexity of the effective action (in terms of φ^n, once ϕ has been eliminated by its equation of motion).

In order to clarify this point we have to construct the k-dependent effective action $\Gamma_k(\varphi)$ from $W_k(J)$ by a Legendre transformation:

$$\Gamma_k(\varphi) = W_k(J) + \int d^4x\, J^n(x)\,\varphi^n(x)\,. \tag{3.25}$$

Taking the functional derivative of (3.25) with respect to $J^n(x)$, and using the expression (3.19) for $W_k(J)$, one obtains for the "classical" field φ^n

$$\varphi^n(x) = -\frac{\delta W_k(J)}{\delta J^n(x)} = \int d^4x'\, P(x,x',\widehat{\phi})\, J^n(x')\,. \tag{3.26}$$

$\widehat{\phi}$ in (3.26) is the stationary point configuration of $W_k(J)$, as given by (3.19), in the presence of sources:

$$\begin{aligned} 0 = \left[\frac{\delta W_k(J)}{\delta\phi(x)}\right]_{\widehat{\phi}} = \Bigg[&\frac{\delta}{\delta\phi(x)}\left(NG_{\text{bare}}(\phi) + N\Delta G(\phi)\right) + \\ &+\frac{1}{2}\int d^4x_1\, d^4x_1'\, d^4x_2\, d^4x_2'\, J^n(x_1)P(x_1,x_1',\phi)\frac{\delta P^{-1}(x_1',x_2',\phi)}{\delta\phi(x)} \times \\ &\times P(x_2',x_2,\phi)\, J^n(x_2)\Bigg]_{\widehat{\phi}}\,. \end{aligned} \tag{3.27}$$

We have expressed the functional derivative of the term $\sim JPJ$ in $W_k(J)$ with respect to ϕ in terms of $\delta P^{-1}/\delta\phi$ for two reasons: first, from (3.15), this functional derivative is trivial:

$$\frac{\delta P^{-1}(x_1',x_2',\phi)}{\delta\phi(x)} = 2\delta^4(x_1'-x)\delta^4(x_2'-x)\,; \tag{3.28}$$

Second, using (3.26) and (3.28), eq. (3.27) becomes very simple once written in terms of φ^n: one finds

$$0 = \left[\frac{\delta}{\delta \phi(x)} \left(N G_{\text{bare}}(\phi) + N \Delta G_{\text{bare}}(\phi) \right) + \varphi^n(x) \varphi^n(x) \right]_{\widehat{\phi}} . \tag{3.29}$$

Eq. (3.29) allows to obtain $\widehat{\phi}$ as a functional of the classical fields $\varphi^n(x)$ or $\mathcal{O}(x) = \varphi^n(x)\varphi^n(x)$. From (3.25) with (3.19) for $W(J)$, the inverse of (3.26) for J^n in terms of φ^n, and (3.15) for P^{-1} one obtains for the effective action $\Gamma_k(\varphi)$:

$$\Gamma_k(\varphi) = S_{kin}(\varphi) + N \left(G_{\text{bare}}(\widehat{\phi}) + \Delta G(\widehat{\phi}) \right) + \int d^4x \; \widehat{\phi}(x) \cdot \mathcal{O}(x) \tag{3.30}$$

where $\widehat{\phi}$ solves (3.29). For the effective potential for $k = 0$ we replace φ^n by constants in (3.30), i.e. we drop $S_{kin}(\varphi)$ and write $\mathcal{O}(x) \to \mathcal{O} \equiv \varphi^n \varphi^n$. Now we proceed to solve (3.29). For constant $\mathcal{O}$ the solution of (3.29) exhibits a similar behaviour in the limit of vanishing infrared cutoff as in (3.24): in the "inner region" of the effective potential, where

$$\mathcal{O} = |\varphi^n|^2 < \frac{m^2}{\lambda} - \frac{\Lambda^2}{16\pi^2} \tag{3.31}$$

(instead of (3.23)), one obtains

$$\begin{aligned} k^2 &\to 0 \, , \\ \widehat{\phi} &\to 0_{-\varepsilon} \, , \\ \frac{1}{8\pi^2} \widehat{\phi} \log \left(\frac{k^2 + 2\widehat{\phi}}{\Lambda^2} \right) &\to \frac{m^2}{\lambda} - \frac{\Lambda^2}{16\pi^2} - \mathcal{O} = \text{finite} \, . \end{aligned} \tag{3.32}$$

Hence, for $|\varphi^n|$ satisfying (3.31) we obtain $\widehat{\phi} = 0$ in the expression (3.30) for the effective potential, which is thus flat (φ-independent).

As mentioned above, the underlying reason for this phenomenon is not the large-N limit, but the convexity of the effective action of φ^n even in the phase with spontaneously broken symmetry. However, its observation requires the introduction of an infrared cutoff k^2, and the study of the limits (3.32).

We discussed this phenomenon in quite some detail, since its essential features are shared by the A_μ-$B_{\mu\nu}$–model, which is the subject of the next subsection.

3.2 The solution of the A_μ-B_μ–model

Now we use a very similar formalism in order to solve the model defined by the partition function (2.17) in the large-N limit. As discussed below (2.17), we may implement both UV cutoffs and infrared cutoffs Λ and k, respectively, with the help of cutoff terms $\Delta S_k(A, B)$ as defined in (2.10) and appropriate properties of the cutoff functions R^A and R^B. (Again we keep the UV cutoff $\Lambda \sim M_W$ fixed, but study the limit $k \to 0$.) Thus we write instead of (2.17)

$$e^{-W_k(J)} = \frac{1}{\mathcal{N}'} \int \mathcal{D}A\,\mathcal{D}B\, e^{-S_{\text{bare}}(F,B) - \Delta S_k(A,B) + \int d^4x \left\{ \frac{1}{2\alpha}(\partial_\mu A^n_\mu)^2 + J^n_{A,\mu} A^n_\mu + J^n_{B,\mu\nu} B^n_{\mu\nu} \right\}} \tag{3.33}$$

with $S_{\text{bare}}(F,B)$ given by (2.16).

Next we introduce auxiliary fields ϕ_i, one for each operator $\mathcal{O}_i$ appearing in the argument of $\mathcal{L}'_{\text{bare}}$ in (2.16) (recall the definitions (2.12) for these bilinear Lorentz scalar operators). In analogy to eq. (3.6) we write

$$e^{-N \int d^4x \mathcal{L}'_{\text{bare}}\left(\frac{\mathcal{O}_i}{N}\right)} = \frac{1}{\mathcal{N}_\phi} \int \mathcal{D}\phi_i \; e^{-NG_{\text{bare}}(\phi_i) - \int d^4x \phi_i \cdot \mathcal{O}_i} \,. \tag{3.34}$$

Again the path integral on the right-hand side of (3.34) can be replaced by its stationary point in the large-N limit, and the relation between G_{bare} and $\mathcal{L}'_{\text{bare}}$ becomes

$$N \int d^4x \; \mathcal{L}'_{\text{bare}} \left(\frac{\mathcal{O}_i}{N} \right) = NG_{\text{bare}}(\phi_i) + \int d^4x \; \phi_i \cdot \mathcal{O}_i \,. \tag{3.35}$$

In contrast to the scalar O(N) model we do not assume here that $\mathcal{L}'_{\text{bare}}$ contains only terms up to quadratic order in $\mathcal{O}_i$. Hence the Legendre transform (3.35) may be quite involved in general, but this is not our point of concern: we may equally define our models right away through functionals $G_{\text{bare}}(\phi_i)$. Note that we allow for derivative terms in $\mathcal{L}'_{\text{bare}}$ where the derivatives act on $\mathcal{O}_i$; under the assumptions of locality specified below eq. (2.9) we will still obtain a local expression for $G_{\text{bare}}(\phi_i)$ including, of course, derivatives acting on ϕ_i.

Replacing (3.34) into (3.33) one finds ($\mathcal{N} = \mathcal{N}' \cdot \mathcal{N}_\phi$)

$$\begin{aligned} e^{-W_k(J)} = & \frac{1}{\mathcal{N}} \int \mathcal{D}\phi_i \int \mathcal{D}A \; \mathcal{D}B \times \\ & \times e^{-NG_{\text{bare}}(\phi_i) - \Delta S_k(A,B)} \times \\ & \times e^{-\int d^4x \left\{ \phi_i \cdot \mathcal{O}_i + \frac{h}{2}(\partial_\mu \tilde{B}^n_{\mu\nu})^2 + \frac{\sigma}{2}(\partial_\mu B^n_{\mu\nu})^2 + \frac{1}{2\alpha}(\partial_\mu A^n_\mu)^2 - J^n_{A,\mu} A^n_\mu - J^n_{B,\mu\nu} B^n_{\mu\nu} \right\}} \end{aligned} \tag{3.36}$$

and the $\mathcal{D}A\mathcal{D}B$ path integrals become again gaussian. The result can formally be written as follows: first we introduce the notation

$$\varphi^n_r, \varphi^n_s = \left\{ A^n_\mu, B^n_{\mu\nu} \right\} , \tag{3.37}$$

i.e. the indices r, s, attached to the fields φ^n denote both the different fields A^n, B^n and the different Lorentz indices. Correspondingly we introduce sources J^n_r, J^n_s for $J^n_{A,\mu}$ and $J^n_{B,\mu\nu}$. Now (3.36) becomes

$$e^{-W_k(J)} = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi_i \; e^{-NG_{\text{bare}}(\phi_i) - N\Delta G(\phi_i) + \frac{1}{2} \int d^4x_1 d^4x_2 \{ J^n_r(x_1) P^{rs}(x_1,x_2,\phi_i) J^n_s(x_2) \}} \tag{3.38}$$

with

$$\Delta G(\phi_i) = -\frac{1}{2}\,\mathrm{Tr}\log\left(P^{rs}(x_1, x_2, \phi_i)\right). \tag{3.39}$$

Again the propagators P^{rs} of the A^n_μ, $B^n_{\mu\nu}$-system are, in principle, proportional to $\delta_{n,m}$ with $n, m = 1, \dots, N$; we omitted these Kronecker symbols for simplicity and took care of the resulting contribution from the trace in (3.39) by the explicit factor N multiplying ΔG in (3.38). The propagators $P^{AA}_{\mu,\nu}$, $P^{AB}_{\mu,\rho\sigma}$ and $P^{BB}_{\mu\nu,\rho\sigma}$ depend on the five terms ΔS_k, $\phi_i \cdot \mathcal{O}_i$, $\frac{h}{2}(\partial \widetilde{B})^2$, $\frac{\sigma}{2}(\partial B)^2$ and $\frac{1}{2}(\partial A)^2$ in the exponent of (3.36). Simple explicit expressions can be obtained only for constant fields ϕ_i. They are given, in the Landau gauge $\alpha \to 0$, in the appendix. For ΔG in (3.39) one obtains for constant ϕ_i

$$\begin{aligned}\Delta G(\phi_i) = \frac{3}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\Big[&\log\Big(\sigma p^2(\phi_1 + R^A(p^2)) + \\ &+ 4(\phi_1 + R^A(p^2))(\phi_3 + R^B(p^2)) - \phi_2^2\Big) + \\ &+ \log\left(hp^2 + 4(\phi_3 + R^B(p^2))\right)\Big]. \end{aligned}\tag{3.40}$$

The cutoff functions R^A and R^B parametrize the simplest form of ΔS_k given in (A.1) in the appendix. Again we will employ, for simplicity, "sharp" cutoffs R^A, R^B which are infinity for $p^2 < k^2$, $p^2 > \Lambda^2$ and zero for $k^2 < p^2 < \Lambda^2$. (We have checked that the stationary point in ϕ_i described by eqs. (3.46) below appears also for smooth cutoff functions R^A, R^B.) Then the d^4p integral in (3.40) can be performed with the result

$$\begin{aligned}\Delta G(\phi_i) = \frac{3}{32\pi^2}\int d^4x \Bigg[&\left(\Lambda^4 - \frac{(4\phi_1\phi_3 - \phi_2^2)^2}{\sigma^2\phi_1^2}\right)\log\left(\Lambda^2\sigma\phi_1 + 4\phi_1\phi_3 - \phi_2^2\right) - \\ &- \left(k^4 - \frac{(4\phi_1\phi_3 - \phi_2^2)^2}{\sigma^2\phi_1^2}\right)\log\left(k^2\sigma\phi_1 + 4\phi_1\phi_3 - \phi_2^2\right) + \\ &+ \frac{4\phi_1\phi_3 - \phi_2^2}{\sigma\phi_1}\left(\Lambda^2 - k^2\right) + \left(\Lambda^4 - \frac{16\phi_3^2}{h^2}\right)\log\left(\Lambda^2 h + 4\phi_3\right) - \\ &- \left(k^4 - \frac{16\phi_3^2}{h^2}\right)\log\left(k^2 h + 4\phi_3\right) + 4\frac{\phi_3}{h}\left(\Lambda^2 - k^2\right)\Bigg]. \end{aligned}\tag{3.41}$$

This expression for $\Delta G(\phi_i)$ has to be inserted into (3.38), and again the $\mathcal{D}\phi_i$ path integral is dominated by its stationary point(s). Hence $W_k(J)$ becomes

$$W_k(J) = NG(\widehat{\phi}_i) - \frac{1}{2}\int d^4x_1\, d^4x_2 \left\{J^n_r(x_1)P^{rs}(x_1, x_2, \widehat{\phi}_i)J^n_s(x_2)\right\} \tag{3.42}$$

where

$$G(\phi_i) = G_{\mathrm{bare}}(\phi_i) + \Delta G(\phi_i) \tag{3.43}$$

and $\widehat{\phi}_i \equiv \widehat{\phi}_i(J)$ satisfy the three equations (recall $i = 1, 2, 3$)

$$\left[\frac{\delta}{\delta\phi_i}\Big(NG(\phi_i) - \frac{1}{2}\int d^4x_1\, d^4x_2\, J^n_r(x_1)P^{rs}(x_1, x_2, \phi_i)J^n_s(x_2)\Big)\right]_{\widehat{\phi}_i(J)} = 0\,. \tag{3.44}$$

First we are interested in the vacuum configurations $\widehat{\phi}_i^0 \equiv \widehat{\phi}_i(J=0)$. The three stationary point equations (3.44) with ΔG given by (3.41) simplify considerably if we switch from the three independent fields ϕ_1, ϕ_2, ϕ_3 to ϕ_1, ϕ_3, Σ with

$$\Sigma = \frac{4\phi_1\phi_3 - \phi_2^2}{\sigma\phi_1}\,. \tag{3.45}$$

Then the stationary point equations for $J=0$ become (a factor N can be dropped)

$$\left[\frac{\delta G_{\mathrm{bare}}}{\delta\phi_1} + \frac{3}{32\pi^2\phi_1}\left(\Lambda^4 - k^4\right)\right]_{\widehat{\phi}_i^0} = 0\,, \tag{3.46a}$$

$$\left[\frac{\delta G_{\mathrm{bare}}}{\delta\phi_3} + \frac{3}{4\pi^2h^2}\left\{4\phi_3\log\left(\frac{4\phi_3+hk^2}{4\phi_3+h\Lambda^2}\right) + h\left(\Lambda^2-k^2\right)\right\}\right]_{\widehat{\phi}_i^0} = 0\,, \tag{3.46b}$$

$$\left[\frac{\delta G_{\mathrm{bare}}}{\delta\Sigma} + \frac{3}{16\pi^2}\left\{\Sigma\log\left(\frac{\Sigma+k^2}{\Sigma+\Lambda^2}\right) + \Lambda^2 - k^2\right\}\right]_{\widehat{\phi}_i^0} = 0\,. \tag{3.46c}$$

Clearly eqs. (3.46) can not be solved without knowledge of G_{bare} (or its Legendre transform $\mathcal{L}_{\mathrm{bare}}$, cf. (3.35)), and they may have several solutions. In the latter case the one with the lowest vacuum energy has to be chosen. This issue, together with the physical interpretation of the auxiliary fields ϕ_i, will be discussed below.

Let us now assume that, in some analogy to (3.23) in the case of the scalar O(N) model,

$$\left.\frac{\delta G_{\mathrm{bare}}}{\delta\Sigma}\right|_{\widehat{\Sigma}^0=0} + \frac{3}{16\pi^2}\Lambda^2 < 0\,. \tag{3.47}$$

Then, provided $\widehat{\Sigma}^0(k^2=\Lambda^2) < 0$, the solution of (3.46c) behaves as follows in the limit of vanishing infra-red cutoff: we have simultaneously

$$k^2 \to 0\,, \tag{3.48a}$$

$$\widehat{\Sigma}^0(k^2) \to 0_{-\varepsilon}\,, \tag{3.48b}$$

$$\frac{3}{16\pi^2}\widehat{\Sigma}^0\log\left(\frac{\widehat{\Sigma}^0+k^2}{\widehat{\Sigma}^0+\Lambda^2}\right) \to -\left.\frac{\delta G_{\mathrm{bare}}}{\delta\Sigma}\right|_{\widehat{\Sigma}^0=0} - \frac{3}{16\pi^2}\Lambda^2\,. \tag{3.48c}$$

From (3.45) the stationary point (3.48b), if it exists, corresponds to (assuming $\widehat{\phi}_1^0<\infty$)

$$4\widehat{\phi}_1^0\,\widehat{\phi}_3^0 - \widehat{\phi}_2^{0^2} = 0\,. \tag{3.49}$$

Eqs. (3.46a) and (3.46b) represent two additional constraints on the three fields $\widehat{\phi}_1^0$, $\widehat{\phi}_2^0$ and $\widehat{\phi}_3^0$, which are thus all determined. As in the scalar O(N) model this stationary point would not have been observed if one puts $k^2=0$ from the start.

Subsequently we will denote the stationary point (3.48), where (3.49) holds, the "confining phase" of the model. As stated above its existence depends on the validity of (3.47); clearly, this condition requires no "fine-tuning" of G_{bare} or $\mathcal{L}_{\mathrm{bare}}$.

Next we construct the effective action of the model. Of course it is given by the Legendre transform of $W_k(J)$ in (3.42):

$$\Gamma_k(A,B) = W_k(J) + \int d^4x \left(J^n_{A,\mu} A^n_\mu + J^n_{B,\mu\nu} B^n_{\mu\nu} \right) . \tag{3.50}$$

Following the steps discussed from eqs. (3.26) to (3.30) in the context of the bosonic O(N) model one obtains

$$\Gamma_k(A,B) = \Delta S_k + NG(\widehat{\phi}_i) + \int d^4x \left(\widehat{\phi}_i \mathcal{O}_i + \frac{h}{2} \left(\partial_\mu \widetilde{B}^n_{\mu\nu} \right)^2 + \frac{\sigma}{2} \left(\partial_\mu B^n_{\mu\nu} \right)^2 \right) \tag{3.51}$$

with $G(\widehat{\phi}_i)$ as in (3.43), and where it is again convenient to replace the sources in the stationary point equations (3.44) by the fields A^n_μ, $B^n_{\mu\nu}$ or the corresponding operators $\mathcal{O}_i$ (2.12):

$$\left[\frac{\delta}{\delta\phi_i} NG(\phi_i) + \mathcal{O}_i \right]_{\widehat{\phi}_i(A,B)} = \left[\frac{\delta\Gamma_k}{\delta\phi_i} \right]_{\widehat{\phi}_i(A,B)} = 0 . \tag{3.52}$$

In the vacuum we require A^n_μ, $B^n_{\mu\nu} = 0$ and hence $\mathcal{O}_i = 0$ in order not to break Lorentz invariance, and then (3.52) coincides with (3.44) for $J = 0$.

Let us briefly discuss the relation between the condition $\widehat{\Sigma}^0(k^2 \to \Lambda^2) < 0$ for the confining phase to appear, and the convexity of the effective action in the case of a non-convex bare action. To this end we consider constant configurations $F^n_{\mu\nu}$, $B^n_{\mu\nu}$ in the expression (3.51) for $\Gamma_k(A,B)$.

Given the dependence of $\mathcal{O}_i$ on $F^n_{\mu\nu}$, $B^n_{\mu\nu}$ in (2.12) it is straightforward to compute the determinant of second derivatives of $\Gamma_k(\mathcal{O}_i)$ with respect to $F^n_{\mu\nu}$, $B^n_{\mu\nu}$. If one drops ΔS_k in (3.51), this determinant is negative if

$$4\widehat{\phi}_1\, \widehat{\phi}_3 - \widehat{\phi}_2^2 < 0 . \tag{3.53}$$

In this case Γ_k without ΔS_k would be non-convex around the origin $F = B = 0$ in field space. Under the assumption $\widehat{\phi}_1 > 0$ (otherwise the term $\sim F^2$ in Γ_k would have the "wrong" sign) (3.53) corresponds to the negativity of $\widehat{\Sigma}$, cf. eq. (3.45). The confining stationary point $\widehat{\Sigma} \to 0_{-\varepsilon}$ for $k^2 \to 0$ corresponds thus to the "ironing" of a non-convex bare action as in the case of a non-convex bare potential in the bosonic O(N) model in the broken phase.

The condition for the reach of the confining phase for $k^2 \to 0$ in the presence of non-vanishing backgrounds A^n_μ, $B^n_{\mu\nu}$ and hence $\mathcal{O}_i$ differs somewhat from the positivity of the right-hand side of (3.48c). After switching to the independent variables ϕ_1, ϕ_3 and Σ, the three equations (3.52) can be written as

$$\left[\frac{\delta}{\delta\phi_1} NG(\phi_i) + \mathcal{O}_1 + \sqrt{\frac{4\phi_3 - \sigma\Sigma}{4\phi_1}}\; \mathcal{O}_2 \right]_{\widehat{\phi}_i} = 0 , \tag{3.54a}$$

$$\left[\frac{\delta}{\delta\phi_3}NG(\phi_i) + \mathcal{O}_3 + \sqrt{\frac{4\phi_1}{4\phi_3 - \sigma\Sigma}}\;\mathcal{O}_2\right]_{\widehat{\phi}_i} = 0\,, \tag{3.54b}$$

$$\left[\frac{\delta}{\delta\Sigma}NG(\phi_i) - \frac{\sigma}{2}\sqrt{\frac{\phi_1}{4\phi_3 - \sigma\Sigma}}\;\mathcal{O}_2\right]_{\widehat{\phi}_i} = 0\,. \tag{3.54c}$$

In principle we can solve (3.54a) and (3.54b) for $\widehat{\phi}_1(\mathcal{O}_i, \widehat{\Sigma})$ and $\widehat{\phi}_3(\mathcal{O}_i, \widehat{\Sigma})$. Then equation (3.54c) has the "confining" solution $\widehat{\Sigma}(k^2) \to O_{-\varepsilon}$ for $k^2 \to 0$ if

$$\left[N\left\{\frac{\delta G_{\text{bare}}(\widehat{\phi}_i)}{\delta\widehat{\Sigma}} + \frac{3}{16\pi^2}\Lambda^2\right\} - \frac{\sigma\widehat{\phi}_1}{2\widehat{\phi}_2}\mathcal{O}_2\right]_{\widehat{\Sigma}=0} < 0\,. \tag{3.55}$$

Notably we then have, for $k^2 \to 0$,

$$4\widehat{\phi}_1\widehat{\phi}_3 - \widehat{\phi}_2^2 = 0 \tag{3.56}$$

even for non-vanishing classical fields A_μ^n, $B_{\mu\nu}^n$ or $\mathcal{O}_i$. Generally, once the right-hand side of (3.48c) is positive, there exists a finite region in the space of operators $\mathcal{O}_i$ where (3.55) holds as well.

Many remarkable properties of the effective action (3.51) in the confining phase (3.56) will be discussed in the next chapter. To close the present section, we will clarify the physical meaning of the auxiliary fields ϕ_i which parametrize the vacua of the model.

To this end we return to the partition function (3.36). Instead of "switching on" sources J for the fields A_μ^n, $B_{\mu\nu}^n$ we will consider sources $K_i(x)$ for the composite operators $\mathcal{O}_i$ and define

$$\begin{aligned} e^{-W_k(K)} = {} & \frac{1}{\mathcal{N}}\int \mathcal{D}\phi_i \int \mathcal{D}A\mathcal{D}B \times \\ & \times e^{-NG_{\text{bare}}(\phi_i) - \Delta S_k(A,B) - \int d^4x\left\{\phi_i\mathcal{O}_i + \frac{h}{2}\left(\partial_\mu \widetilde{B}_{\mu\nu}^n\right)^2 + \frac{\sigma}{2}\left(\partial_\mu B_{\mu\nu}^n\right)^2 + \frac{1}{2\alpha}\left(\partial_\mu A_\mu^n\right)^2 - K_i\mathcal{O}_i\right\}}\,. \end{aligned} \tag{3.57}$$

(We could have kept the sources J; we have omitted them just for simplicity.) Evidently $W_k(K)$ generates the Green functions of the composite operators $\mathcal{O}_i$; notably we have

$$\langle\mathcal{O}_i(x)\rangle_K = -\frac{\delta W_k(K)}{\delta K_i(x)}\,. \tag{3.58}$$

One observes in (3.57) that the sources K_i couple to the bilinear operators $\mathcal{O}_i$ in the same way as the auxiliary fields ϕ_i. Repeating the steps which have led to eq. (3.42) for $W_k(J)$ one thus obtains immediately

$$W_k(K) = NG_{\text{bare}}(\widehat{\phi}_i) + N\Delta G\left(\widehat{\phi}_i - K_i\right) \tag{3.59}$$

with ΔG as in (3.41), and where the fields $\widehat{\phi}_i \equiv \widehat{\phi}_i(K)$ satisfy

$$\left[\frac{\delta}{\delta\phi_i}\Big(G_{\rm bare}(\phi_i) + \Delta G\left(\phi_i - K_i\right)\Big)\right]_{\widehat{\phi}_i(K)} = 0\,. \tag{3.60}$$

Using (3.59) and (3.60) in (3.58) one finds

$$\langle \mathcal{O}_i(x)\rangle_K = N\frac{\delta\Delta G(\widehat{\phi}_i - K_i)}{\delta\widehat{\phi}_i(x)}\,. \tag{3.61}$$

Hence in the classical limit (or for an infrared cutoff $k^2 \to \Lambda^2$) where ΔG vanishes, the vacuum expectation values of the composite operators $\mathcal{O}_i$ vanish as they should.

In the limit of vanishing source K_i we can use (3.60) again to write

$$\langle \mathcal{O}_i(x)\rangle_{K=0} = -N\frac{\delta G_{\rm bare}(\widehat{\phi}_i)}{\delta\widehat{\phi}_i} \tag{3.62}$$

where the auxiliary fields $\widehat{\phi}_i$ satisfy the stationary point equation (3.60) with K_i=0. At first sight (3.62) seems to be a trivial consequence of the Legendre transform (3.35), but now the $\widehat{\phi}_i$ are the extrema of $G_{\rm bare} + \Delta G$ including the quantum contribution. In any case (3.62) shows that different vevs $\widehat{\phi}_i$ parametrize different vevs $\langle\mathcal{O}_i\rangle$ (recall their definitions (2.12)). This relation would be linear if $G_{\rm bare}$ would be quadratic in ϕ_i (as in the renormalizable O(N) model).

Since the ϕ_i are auxiliary fields for composite operators the interpretation of the effective action (3.51) (at $k = 0$, where ΔS_k vanishes, and for constant field configurations) as an energy density is not straightforward [20], if the ϕ_i are considered as independent variables. Notably $\Gamma(\phi_i, A, B)$ is generically unbounded from below for $|\phi_i| \to \infty$ due to the negativity of $G_{\rm bare}(\phi_i)$. (Such a feature of a potential V involving auxiliary fields is well-known from supersymmetric theories, once auxiliary F- or D-fields are introduced in order to complete the supermultiplets. Then one has e.g., $V(F,\varphi) = -|F_i|^2 + F_iW_i + \text{h.c.}$ where W_i is the derivative of the superpotential $W(\varphi_i)$). At the stationary points $\phi_i = \widehat{\phi}_i$, however, the interpretation of $\Gamma(\widehat{\phi}_i, A = B = 0)$ as an energy density can be maintained [20] and the energy densities of different vacua can be compared.

The interesting properties of the effective action in the confining phase with (3.49) will be the subject of the next section.

4. Properties of the confining phase

In the following we assume that the necessary condition (3.55) for the existence of a confining phase is satisfied within a finite regime of field configurations A^n_μ, $B^n_{\mu\nu}$

(and corresponding bilinears $\mathcal{O}_i$), and that it represents the lowest (or only) vacuum of the model. For $k^2 \to 0$ the effective action (3.51) reads

$$\Gamma(A,B) = NG(\widehat{\phi}_i) + \int d^4x \left\{ \widehat{\phi}_i \mathcal{O}_i + \frac{h}{2}\left(\partial_\mu \widetilde{B}^n_{\mu\nu}\right)^2 + \frac{\sigma}{2}\left(\partial_\mu B^n_{\mu\nu}\right)^2 \right\} \tag{4.1}$$

with $G(\phi_i)$ as in (3.43), and the $\widehat{\phi}_i$ satisfy (3.52). The confining phase is characterized by the validity of eq. (3.56), $4\widehat{\phi}_1\widehat{\phi}_3 - \widehat{\phi}_2^2 = 0$.

First we consider a new kind of N U(1) gauge transformations which involve vector-like gauge parameters Λ^n_μ [9]:

$$\begin{aligned} &\delta A^n_\mu(x) = \Lambda^n_\mu(x)\,, \qquad \delta F^n_{\mu\nu}(x) = \partial_\mu \Lambda^n_\nu(x) - \partial_\nu \Lambda^n_\mu(x) \equiv \Lambda^n_{\mu\nu}(x)\,, \\ &\delta B^n_\mu(x) = \sqrt{\frac{\widehat{\phi}_1}{\widehat{\phi}_3}}\, \Lambda^n_{\mu\nu}(x)\,. \end{aligned} \tag{4.2}$$

Using eq. (3.56) in order to eliminate $\widehat{\phi}_2$, (choosing $\widehat{\phi}_2$ positive, otherwise the sign in δB has to be changed) and (2.12) for the definitions of the bilinear $\mathcal{O}_i$, one finds that the terms $\widehat{\phi}_i\mathcal{O}_i$ in (4.1) can be written as

$$\widehat{\phi}_i \mathcal{O}_i = \sum_n \left(\sqrt{\widehat{\phi}_1}\, F^n_{\mu\nu} + \sqrt{\widehat{\phi}_3}\; B^n_{\mu\nu} \right)^2 . \tag{4.3}$$

Hence these terms are invariant under the gauge transformations (4.2). Next we have a look at the term $\sim (\partial_\mu \widetilde{B}^n_{\mu\nu})^2$. Thanks to a Bianchi identity one obtains

$$\delta\left(\partial_\mu \widetilde{B}^n_{\mu\nu}\right)^2 = 0 + \mathcal{O}\left(\partial_\mu \widehat{\phi}_i\right), \tag{4.4}$$

i.e. it is also invariant up to derivatives acting on $\widehat{\phi}_i$ which appear in the variation $\delta B^n_{\mu\nu}$ in (4.2). Recall that the $\widehat{\phi}_i$ are fixed in terms of the bilinear operators $\mathcal{O}_i$ by (3.52); for constant configurations $\mathcal{O}_i$ the $\widehat{\phi}_i$ would also be constant, and the symmetry breaking contribution to (4.4) is thus proportional to $\partial_\mu \mathcal{O}_i$.

The last term $\sim (\partial_\mu B^n_{\mu\nu})^2$ in (4.1) behaves as a gauge fixing term of the symmetry (4.2), and altogether one obtains a Ward identity

$$\delta\Gamma(A,B) = -\sqrt{\frac{\widehat{\phi}_1}{\widehat{\phi}_3}}\, \sigma \left(\partial_\mu B^n_{\mu\nu}\right)\left(\partial_\mu \Lambda^n_{\mu\nu}\right) + \mathcal{O}\left(\partial_\mu \mathcal{O}_i \cdot (\partial_\mu B)^2\right). \tag{4.5}$$

It is natural to expect that the symmetry (4.2) is broken by higher derivative terms (beyond the gauge fixing term): the bare action S_{bare} (2.16) of the model does certainly not exhibit the symmetry (4.2) (or satisfy the Ward identity (4.5)), and the Green functions at large non-exceptional euclidean momenta with $p^2 \to \Lambda^2$ are generated by S_{bare}. This fact is realized by the dependence of the effective action on higher derivative terms. The symmetry (4.2) is thus a pure "low energy" phenomenon.

Note, however, that the “violation” of the Ward identity (4.5) is of higher order in the fields, and is proportional to derivatives acting on the bilinears $\mathcal{O}_i$. Hence the symmetry (4.2) is realized on modes A^n_μ and $B^n_{\mu\nu}$ which may be rapidly oscillating, but correspond to constant bilinear configurations $\mathcal{O}_i$. (These modes are associated with Green functions with “exceptional” external momenta.)

The implication of the gauge symmetry (4.2) on these modes of the U(1) gauge fields A^n_μ is evidently that they can be “gauged away” and “eaten” by the (massive or even infinitely massive) Kalb-Ramond fields $B^n_{\mu\nu}$ [9].

Correspondingly the (approximate) gauge symmetry (4.2) will have consequences on the “response” of the model with respect to couplings to external sources J^A and J^B. Before studying this issue we will discuss the behaviour of the effective action (4.1) under a duality transformation, whose existence is closely related to the symmetry (4.2).

Let us recall the essential features of a duality transformation: it corresponds to the introduction of dual fields such that the Bianchi identities associated to the dual fields correspond to the equations of motion of the “original” fields (up to gauge fixing terms), and the Bianchi identities associated to the “original” fields to the equations of motion of the dual fields.

Using (4.3), the equations of motion for A^n_μ and $B^n_{\mu\nu}$ are

$$\partial_\mu\left(\widehat{\phi}_1 F^n_{\mu\nu} + \sqrt{\widehat{\phi}_1\widehat{\phi}_3}\,B^n_{\mu\nu}\right) = 0\,, \qquad (4.6a)$$

$$2\sqrt{\widehat{\phi}_1\widehat{\phi}_3}\,F^n_{\mu\nu} + 2\widehat{\phi}_3 B^n_{\mu\nu} - \frac{h}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\rho\partial_\lambda\widetilde{B}^n_{\lambda\sigma} - \frac{\sigma}{2}\left(\partial_\mu\partial_\sigma B^n_{\sigma\nu} - \partial_\nu\partial_\sigma B^n_{\sigma\mu}\right) = 0\,. \qquad (4.6b)$$

As dual fields we introduce vectors C^n_μ (whose field strength tensor will be denoted by $F^{c,n}_{\mu\nu}$), and pseudoscalars φ^n. They are related to the original fields A^n_μ, $B^n_{\mu\nu}$ through

$$\frac{1}{2}\,F^{c,n}_{\mu\nu} = \widehat{\phi}_1\widetilde{F}^n_{\mu\nu} + \sqrt{\widehat{\phi}_1\widehat{\phi}_3}\,\widetilde{B}^n_{\mu\nu}\,, \qquad (4.7a)$$

$$\partial_\mu\varphi^n + C^n_\mu = \frac{h}{2}\sqrt{\frac{\widehat{\phi}_1}{\widehat{\phi}_3}}\,\partial_\mu\widetilde{B}^n_{\mu\nu} \qquad (4.7b)$$

(recall our definition of $\widetilde{B}^n_{\mu\nu}$ etc. in eq. (2.14)).

The Bianchi identities associated to the dual fields are

$$\partial_\mu\widetilde{F}^{c,n}_{\mu\nu} = 0\,, \qquad (4.8a)$$

$$\varepsilon_{\mu\nu\rho\sigma}\,\partial_\rho\partial_\sigma\varphi^n = 0\,. \qquad (4.8b)$$

Contracting the dual of (4.7a) with ∂_μ one realizes immediately that the Bianchi identity (4.8a) coincides with the equation of motion (4.6a). The Bianchi identity (4.8b) should reproduce the equation of motion (4.6b) up to the “gauge fixing term” $\sim\sigma$. Indeed this is the case up to terms proportional to $\partial_\mu\widehat{\phi}_i$.

The Bianchi identities associated with the original fields A^n_μ, $B^n_{\mu\nu}$ read

$$\partial_\mu \widetilde{F}^n_{\mu\nu} = 0\,, \tag{4.9a}$$

$$\partial_\mu \partial_\nu \widetilde{B}^n_{\mu,\nu} = 0\,. \tag{4.9b}$$

As action for the dual fields C^n_μ, φ^n we propose (up to gauge fixing terms)

$$\Gamma_{Dual} = \int d^4x \left\{ \frac{1}{4\sqrt{\widehat{\phi}_1 \widehat{\phi}_3}} F^{c,n}_{\mu\nu}\, F^{c,n}_{\mu\nu} + \frac{2}{h}\sqrt{\frac{\widehat{\phi}_3}{\widehat{\phi}_1}} \left(\partial_\mu \varphi^n + C^n_\mu\right)^2 \right\}. \tag{4.10}$$

Now one finds, using (4.7b), that the Bianchi identity (4.9b) coincides exactly with the equation of motion for φ^n derived from (4.10), whereas the Bianchi identity (4.9a) agrees with the equation of motion of C^n_μ only up to terms $\sim \partial_\mu \widehat{\phi}_i$.

Not astonishingly, duality is thus realized to the same level as the gauge symmetry (4.2): to the order quadratic in the fields the physical content of the dual action (4.10) is the same as the original effective action (4.1), since the "perturbing" terms $\sim \partial_\mu \widehat{\phi}_i$ are effectively of higher order. (As solutions of the stationary point equations (3.52) the deviations of the fields $\widehat{\phi}_i$ from their constant values in the vacuum are proportional to the bilinear operators $\mathcal{O}_i$.) To higher order in the fields duality is realized only on modes of A^n_μ, $B^n_{\mu\nu}$ which correspond to constant bilinears $\mathcal{O}_i$.

The physical interpretation of the dual action (4.10) is obviously the one of an abelian $\mathrm{U}(1)^N$ Higgs model in the spontaneously broken phase where φ^n represent the Goldstone bosons, and where the gauge fields C^n_μ have acquired a mass $2(\widehat{\phi}_3/h)^{1/2}$. Since this represents the "low energy effective action" of a theory in which the "dual" electric charge has condensed in the vacuum, the original action (4.1) corresponds to the situation where the "magnetic" charge has condensed in the vacuum.

Let us return to the response of our model with respect to external sources. The expression for $W(J)$ has been given in eq. (3.42) in the preceeding section, and subsequently we consider the limit of vanishing infrared cutoff $k^2 \to 0$. We recall that the fields $\widehat{\phi}_i(J)$ satisfy the stationary point equations (3.44), and satisfy thus obviously

$$\widehat{\phi}_i(J) = \widehat{\phi}^0_i + \mathcal{O}(J^2) \tag{4.11}$$

(where the $\widehat{\phi}^0_i$ satisfy (3.49)).

In fact, the procedure to solve (3.44) exactly would be quite cumbersome, since one would need the expressions for the propagators $P^{rs}(x_1, x_2, \phi_i)$ for arbitrary ϕ_i. In practice it is much easier to work with the effective action $\Gamma(A, B)$ (3.51); this procedure will be described below.

On the other hand, if the sources J are sufficiently weak, we can expand $W(J)$ in powers of J. Since the $\widehat{\phi}^0_i$ are stationary points of $G(\phi_i)$ we have

$$W(J) = NG(\widehat{\phi}^0_i) - \frac{1}{2}\int d^4x_1\, d^4x_2 \left\{ J^n_r(x_1)\ P^{rs}(x_1, x_2, \widehat{\phi}^0_i) J^n_s(x_2) \right\} + \mathcal{O}(J^4)\,. \tag{4.12}$$

Interesting informations can already be obtained from the second term of $\mathcal{O}(J^2)$ in eq. (4.12), which involves the propagators P^{rs} in a confining vacuum given in eqs. (A.7) and (A.8) in the appendix.

Let us start with a source $J^n_{A,\mu}(x)$ for the fields A^n_μ only. The simplest geometrical configuration is a Wilson loop source $J^n_{A,\mu}$ which is non-vanishing only on a curve C embedded in 4d space-time, where C has to be closed because of current conservation:

$$J^n_{A,\mu}(x) = ig_A \int_C dx'_\mu \, \delta^4(x - x') \, . \tag{4.13}$$

The term quadratic in J in (4.12) then becomes

$$\frac{Ng_A^2}{2} \int_C dx_{1,\mu} \int_C dx_{2,\nu} \, P^{AA}_{\mu,\nu}(x_1 - x_2) \tag{4.14}$$

with P^{AA} as in eq. (A.7). The divergent constants in (A.7) disappear actually in the expression (4.14). This can be seen most easily by applying the (abelian) Gauss law, and expressing the source $J^n_{A,\mu}$ in terms of a source $J^n_{F,\mu\nu}$ which is non-vanishing on an (arbitrary) surface S bounded by the contour C:

$$J^n_{A,\mu}(x) = 2\partial_\nu J^n_{F,\nu\mu}(x) \tag{4.15}$$

with

$$J^n_{F,\mu\nu}(x) = ig_F \int_S d^2\sigma_{\mu\nu}(z) \, \delta^4(x - z) \tag{4.16}$$

where $g_F = g_A/2$, z parametrizes the surface S and

$$d^2\sigma_{\mu\nu}(z) = \varepsilon_{\mu\nu\rho\sigma} \, n^1_\rho(z) \, n^2_\sigma(z) \; d^2 z \, . \tag{4.17}$$

n^1_ρ, n^2_σ are two orthogonal unit vectors perpendicular to S.

The expression (4.14) then turns into

$$2Ng_F^2 \int_S d^2\sigma_{\mu\nu}(z) \int_S d^2\sigma_{\rho\sigma}(z') \, P^{FF}_{\mu\nu,\rho\sigma}(z - z') \tag{4.18}$$

with P^{FF} as in eq. (A.8a). The dependence on the divergent constants in eq. (A.7) has disappeared as announced.

In the limit where the surface S becomes very large one finds that the expression (4.18) is proportional to the (minimal) surface bounded by the contour C which we will also denote by S.

Thus we have obtained the area law for the expectation value of the Wilson loop. This result has to be swallowed, however, with two grains of salt. First, in the limit where S and hence the expression (4.18) become large, it becomes inconsistent to confine oneself to terms of $\mathcal{O}(J^2)$ in $W(J)$. (Fortunately, as we will discuss below, a more complete treatment just modifies the contributions to the string tension, not the area law.)

Second, and more importantly, an inconsistency arises if we couple our model to a quantum field theory. In order to clarify this inconsistency we first consider a slightly modified configuration of sources J_F: let us consider two distinct "Wilson surfaces" S_1, S_2, which may both be small and centered at x_1, x_2, respectively. Accordingly the source J_F becomes the sum of two terms, $J_F = J_{F,1} + J_{F,2}$, where $J_{F,1}$ and $J_{F,2}$ are non-vanishing only near x_1 or x_2. Inserting J_F into the term quadratic in J in $W(J)$ in (4.12) one obtains, apart from "self-contractions", a mixed term which is of the order of $P^{FF}_{\mu\nu,\rho\sigma}(x_1 - x_2)$ for $|x_1 - x_2|$ large compared to the diameters of the Wilson surfaces. (The Lorentz indices depend on the orientations of the surfaces S_1 and S_2.) Due to the second term in (A.8a) $P^{FF}(x_1 - x_2)$ decreases only as $|x_1 - x_2|^{-2}$ for generic relative orientations.

The resulting expression can be interpreted as a contribution to the action of the configuration: the effective action Γ is given by the Legendre transform (3.50), and up to $\mathcal{O}(J^4)$ the expression for Γ thus coincides with the expression (4.12) for W up to a change of the sign of the second term.

At this point the result is not yet problematic. Let us now consider the coupling of our model to a quantum field theory in the form of $\widetilde{J}^n_{F,\mu\nu} F^n_{\mu\nu}$, where the source term $\widetilde{J}^n_{F,\mu\nu}$ is quadratic in "external" quantum fields. (SU(N_c) Yang-Mills theory, e.g., contains couplings of the form $f_{nab} F^n_{\mu\nu} W^a_\mu W^b_\nu$, which corresponds to $\widetilde{J}^n_{F,\mu\nu} = f_{nab} W^a_\mu W^b_\nu$.) Vacuum bubbles of the quantum fields then correspond to a "background" of sources $\widetilde{J}^n_{F,\mu\nu}$, which fills the whole space-time (to be averaged over $\widetilde{J}^n_{F,\mu\nu}$ with, e.g., a gaussian measure). If the contribution to the action due to the induced interactions between two vacuum bubbles decreases only as the (distance)$^{-2}$ as described above, space-time filled with vacuum bubbles will lead to an infinite action. Even more severely, every additional Wilson loop switched on "by hand" will also lead to an additional divergent contribution to the action due to its interactions with the vacuum bubbles (after averaging over $\widetilde{J}^n_{F,\mu\nu}$ this divergence is only logarithmic at large distances). Hence it seems to be disallowed.[1]

Actually, before we have assumed that we have obtained our model as an effective low energy theory *after* the fields W^a_μ have been integrated out. The present paradox persists, however, if previously we have only integrated over W^a_μ-modes with $p^2 \gtrsim k_W^2$ (with k_W^2 as small as one likes, but $\neq 0$), and the remaining modes with $p^2 \lesssim k_W^2$ generate the vacuum bubbles described above. In addition, in full QCD we have more fields like quarks, whose vacuum bubbles will generate a background density of sources $\widetilde{J}^n_{F,\mu\nu}$.

The fact that we assumed that W^a_μ-modes with $p^2 \gtrsim k_W^2$ have already been integrated out will, on the other hand, lead to a solution of the problem: first, in the

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[1] Note the important difference to ordinary QED: here the contribution to the euclidean action induced by two vacuum bubbles, i.e. two dipoles localized both in space and time, decreases as the (distance)$^{-4}$. "Switching on" an additional single dipole immersed in a bath of disoriented dipoles then only costs a finite amount of action.

original SU(N_c) Yang-Mills theory, a gauge-invariant Wilson loop "source" is coupled both to A^n_μ and to W^a_μ. If we integrate over W^a_μ-modes after having introduced the fields $B^n_{\mu\nu}$ as in eqs. (2.5) and (2.6), this will necessarily generate a coupling of the Wilson loop source to $B^n_{\mu\nu}$. Hence it is inconsistent to switch on a source for $F^n_{\mu\nu}$, but not simultaneously a source for $B^n_{\mu\nu}$. The simplest expression for a source for $B^n_{\mu\nu}$ is again of the form (4.16)

$$J^n_{B,\mu\nu}(x) = ig_B \int_S d^2\sigma_{\mu\nu}(z)\, \delta^4(x-z)\,, \tag{4.19}$$

where the surface S coincides with the one in (4.16). The value of the coupling constant g_B can, a priori, not be predicted within our model (neither the couplings g_F or g_A in (4.16) or (4.13)).

Thus we return to the term quadratic in J in (4.12), and insert for the source J a sum $J_F + J_B$ with J_F as in (4.16), and J_B as in (4.19). With the propagators from (A.7) and (A.8) this term becomes

$$\frac{1}{16\pi^2\sigma}\int_S d^2\sigma_{\mu\nu}(z_1)\int_S d^2\sigma_{\rho\sigma}(z_2)\Bigg\{\left(\sqrt{\frac{\phi_3}{\phi_1}}g_F - g_B\right)^2 T_{1,\mu\nu,\rho\sigma}(\partial_{z_1})\log|z_1-z_2| + \\ + \mathcal{O}\left(|z_1-z_2|^{-4}\right)\Bigg\}\,. \tag{4.20}$$

Thus the terms of $\mathcal{O}(|z_1-z_2|^{-2})$ in the curled parenthesis in (4.20) vanish if

$$\sqrt{\phi_3}\, g_F = \sqrt{\phi_1}\, g_B\,. \tag{4.21}$$

Again, at this point this is not an obligatory constraint. However, if we switch on a) "Wilson loop" sources J_F, J_B, b) simultaneously uncorrelated "background sources" $\widetilde{J}_F$, $\widetilde{J}_B$, c) average over $\widetilde{J}_F$, $\widetilde{J}_B$ with an arbitrary gaussian measure, we obtain a logarithmically infrared divergent expression, unless the condition (4.21) on J_F, J_B is satisfied. Hence, in the background of a fluctuating vacuum it costs infinite action to turn on sources J_F, J_B which are not related through (4.21). This constraint can be interpreted as a constraint on the ratio ϕ_1/ϕ_3 in the presence of sources, or on the ratio g_F/g_B (once g_F and g_B vary, e.g., with the vacuum configuration of a "microscopic" theory) or, most realistically, on a combination of both.

Before discussing the resulting behaviour of the expectation value of the Wilson loop we note that the above consideration leads, in general, to the following constraint on sources J_A, J_B or J_F, J_B:

$$\sqrt{\phi_3}\, J^n_{A,\mu}(x) = 2\sqrt{\phi_1}\,\partial_\nu J^n_{B,\nu\mu}(x) \quad \text{or} \tag{4.22a}$$

$$\sqrt{\phi_3}\, J^n_{F,\mu\nu}(x) = \sqrt{\phi_1}\, J^n_{B,\mu\nu}(x)\,. \tag{4.22b}$$

The constraints (4.22) are obviously related to the gauge symmetry (4.2): if we replace ϕ_i by the vacuum configurations (in the absence of sources) $\widehat{\phi}_i$, (4.22) corresponds to the "current conservation condition", i.e. to the condition that

$$\int d^4x \left(J^n_{A,\mu}\, A^n_\mu + J^n_{B,\mu\nu}\, B^n_{\mu\nu}\right) \tag{4.23}$$

is invariant under (4.2).

In the case of "conventional" gauge symmetries the current conservation conditions can (and have to) be imposed by hand in order to ensure renormalizability and unitarity of the theory. In the present model, on the one hand, current conservation cannot be imposed from the beginning, since the associated (approximate) gauge symmetry appears only at the level of the effective action once the equations of motion of the fields ϕ_i are satisfied. On the other hand, renormalizability is not an issue here, since we consider an effective low energy theory with a fixed UV cutoff. Also unitarity is trivial as long as we assume the absence of poles in the propagators in the minkowskian regime, cf. the discussion before eq. (2.15). Thus the model has no S-matrix at all, i.e. no asymptotic states. (The absence of bound states will briefly be discussed below). Consequently at this level no constraints on the sources arise.

Only if we couple the model to a quantum field theory we have to reconsider the question of unitarity, i.e. the possibility to project — in a Lorentz covariant way — onto a positive semi-definite part of its Hilbert space which is represented, loosely speaking, by our sources J viewn as functionals of fields. Precisely in this situation "vacuum bubbles" impose eqs. (4.22), the analog of current conservation in an abelian Higgs model with an additional Lorentz index attached to the currents.

If eqs. (4.22) hold the expectation value of the Wilson loop has to be reconsidered: the Wilson loop corresponds now to a source J_A of the form (4.13), which can be rewritten as a source J_F of the form (4.16) using (4.15), *plus* a source J_B of the form (4.19) where the surface S coincides with the one in (4.16).

First, the term quadratic in J in (4.12) consists now only of the "short range" contributions neglected in (4.20). Using again the propagators (A.7) and (A.8) these read

$$\begin{aligned} &\int_S d^2\sigma_{\mu\nu}(z_1) \int_S d^2\sigma_{\rho\sigma}(z_2) \frac{-g_F^2}{16\pi^2\phi_1}\sqrt{\frac{\phi_3}{h}} \times \\ &\times \left(T_{1,\mu\nu,\rho\sigma}(\partial) - \frac{4\phi_3}{h} T_{2,\mu\nu,\rho\sigma}\right) \frac{1}{|z_1 - z_2|} K_1\left(2|z_1 - z_2|\sqrt{\frac{\phi_3}{h}}\right). \end{aligned} \tag{4.24}$$

With (4.21) the contributions $\sim |z_1 - z_2|^{-4}$ cancel as well, and we recall that the tensors T_1 and T_2 are given in (A.2) in the appendix.

In the limit where the surface S becomes large the expression (4.24) behaves as

$$S \cdot \frac{2g_F^2}{\pi\phi_1}\left(\frac{\phi_3}{h}\right)^{3/2} \int_0^\infty dz\, K_1\left(2z\sqrt{\frac{\phi_3}{h}}\right). \tag{4.25}$$

Hence it implies the area law in spite of the cancellations of the long range contributions of the propagators. (Since we have omitted the UV cutoffs in the space-time propagators the expression (4.25) is seemingly UV divergent.)

Expressions of the form (4.25) for the (negative) exponent of the expectation value of the Wilson loop have already appeared repeatedly in the literature in the context of the method of vacuum correlators [3, 22]: in the gaussian approximation (the stochastic vacuum model) the expectation value of the Wilson loop is given by the expectation value of the field strength correlator, and many models for this correlator lead to (4.25) [6, 16]. Actually already the previous result (4.18) — the area law for Wilson loop sources J_A only — has an interpretation in this approach: it would correspond to a function $D_1(x^2)$ in the standard decomposition of the field strength correlator [3, 6, 22] which decreases only as $|x|^{-2}$ for large $|x|$; such a behaviour also implies the area law, but it is strongly disfavoured both phenomenologically [5] and from lattice data [23].

As we have already stated several times above, sources for operators in a $\mathrm{SU}(N_c)$ Yang-Mills theory correspond in our model — if considered as an "effective low energy theory" — to a priori unknown superpositions of sources for A^n_μ (or $F^n_{\mu\nu}$) and $B^n_{\mu\nu}$. The previous discussion leading to eqs. (4.22) fixes this ambiguity, and precisely in this case the two-point correlators decrease exponentially in agreement with the $\mathrm{SU}(N_c)$ lattice data.

In our model the expression (4.25) for the resulting string tension can, however, not be taken too seriously, since in the limit of a large surface S the terms of higher order in J in $W(J)$ in (4.12) can no longer be neglected.

In order to compute $W(J)$ in the "non-linear" regime one has to start first with the effective action $\Gamma(A, B)$ as given in eq. (3.51) (in the limit of vanishing infrared cutoff k) or eq. (4.1). One has to solve the combined equations of motion for A^n_μ, $B^n_{\mu\nu}$ and ϕ_i in the presence of sources $J^n_{A,\mu}$ and $J^n_{B,\mu\nu}$: the corresponding equations for A^n_μ and $B^n_{\mu\nu}$ are given by eqs. (4.6) (where, however, the zeros on the right-hand sides have to be replaced by the corresponding source terms and $2\sqrt{\widehat{\phi}_1\widehat{\phi}_3}$ has to be replaced by $\widehat{\phi}_2$). The equations for ϕ_i are given by eqs. (3.52) or (3.54), where the operators $\mathcal{O}_i$ depend on $F^n_{\mu\nu}$ and $B^n_{\mu\nu}$ as in (2.12).

The corresponding fields have to be inserted into $\Gamma(A, B)$ in (4.1), and then one has to "undo" the Legendre transformation (3.48) in order to obtain $W(J)$. As already noted this last step is actually trivial since $\Gamma(A, B)$ is quadratic in A, B: it suffices to change the sign of the second term in the expression (4.1). Then one can study the dependence of $W(J)$ on J in its full beauty.

Albeit generally the solution of the combined equations of motion is certainly quite involved, there is one situation where it becomes straightforward: let us assume that we have turned on sources $J^n_{F,\mu\nu}(x)$ and $J^n_{B,\mu\nu}(x)$ which are related as in eq. (4.22b), and which are constant inside a space-time volume V, and vanishing outside. Let us furthermore assume that we can neglect all derivative terms in $G(\phi_i)$. Then we can allow for constant configurations $\widehat{\phi}_i^V$ inside the volume V, and constant (generally different) configurations $\widehat{\phi}_i^0$ outside V; the discontinuities at the boundaries of V cost no energy in this case. The arguments leading to eqs. (4.22) are based on the long distance behaviour of the propagators in the vacuum outside V, accordingly eqs. (4.22) have to hold for $\widehat{\phi}_i^0$ outside V. For simplicity we assume that they also hold for $\widehat{\phi}_i^V$ inside V, i.e. that $\widehat{\phi}_1^V/\widehat{\phi}_3^V = \widehat{\phi}_1^0/\widehat{\phi}_3^0$.

For the fields $F^n_{\mu\nu}(x)$ and $B^n_{\mu\nu}(x)$ the following simple solution is then possible: for $B^n_{\mu\nu}(x)$ one chooses

$$B^n_{\mu\nu}(x) = 0 \tag{4.26}$$

everywhere, both inside and outside V (because of the derivatives acting on $B^n_{\mu\nu}$ in the action discontinuities in B are not allowed). Then eqs. (4.6) — with source terms included and using the "confining" relation $\phi_2 = 2\sqrt{\phi_1\phi_3}$ both inside V and outside V — collapse to

$$2\partial_\mu\Big(\phi_1 F^n_{\nu\mu}(x)\Big) = \partial_\mu J^n_{F,\mu\nu}(x)\,, \tag{4.27a}$$

$$2\sqrt{\phi_1\phi_3}F^n_{\mu\nu}(x) = J^n_{B,\nu\mu}(x)\,. \tag{4.27b}$$

With J_F and J_B are related as in (4.22b) inside V eqs. (4.27) are trivially solved simultaneously by

$$F^n_{\mu\nu}(x) = \frac{1}{\widehat{\phi}_1^V}\, J^n_{F,\mu\nu}(x) \qquad \text{inside } V\,, \tag{4.28a}$$

$$F^n_{\mu\nu}(x) = 0 \qquad \text{outside } V \text{ (where } J_F = J_B = 0)\,. \tag{4.28b}$$

Hence, in this simple scenario the effective action (4.1) is given by

$$\Gamma = \int_V d^4x \left\{N\widetilde{G}(\widehat{\phi}_i^V) + \widehat{\phi}_1^V \mathcal{O}_1\right\} + \int_{\bar{V}} d^4x\, N\widetilde{G}(\widehat{\phi}_i^0)\,, \tag{4.29}$$

where $\widetilde{G}(\phi_i)$ is the density associated to $G(\phi_i)$, and $\bar{V}$ denotes the entire space-time volume outside V. (Recall that only $\mathcal{O}_1$ is non-zero for $B^n_{\mu\nu} = 0$.) Since the configurations $\widehat{\phi}_i^0$ are local maxima of $\widetilde{G}(\phi_i)$ and the configurations $\widehat{\phi}_i^V$ are local maxima of $\widetilde{G}(\phi_i) + \phi_1\mathcal{O}_1$ one derives easily

$$N\widetilde{G}(\widehat{\phi}_i^V) + \widehat{\phi}_1^V\mathcal{O}_1 > N\widetilde{G}(\widehat{\phi}_i^0)\,. \tag{4.30}$$

Hence the presence of constant sources inside a volume V increases the euclidean action by an amount which is proportional to V (in an approximation where derivative terms in $G(\phi_i)$ are neglected).

This consideration can be applied to the Wilson loop — i.e. sources J_F, J_B given by (4.16) and (4.19) with g_F and g_B related by $\sqrt{\widehat{\phi}_3^V}g_F = \sqrt{\widehat{\phi}_1^V}g_B$ — provided we replace the enclosed surface S by a "layer" of finite thickness: in the presence of an UV cutoff Λ we cannot resolve distances smaller than Λ^{-1} anyhow, and hence we should replace the surface S in (4.16) and (4.19) by a layer of thickness Λ^{-1} and a corresponding volume $V = S \cdot \Lambda^{-1}$.

Given the above discussion one then obtains immediately an action Γ which is proportional to S, and — after a Legendre transform — $W(J)$ proportional to S and thus the area law. In the present scenario the profile of the flux tube associated to the Wilson loop would be discontinuous, with $\phi_i = \widehat{\phi}_i^V$ (and $F^n_{\mu\nu}$ given by (4.28a)) inside the flux tube, and $\phi_i = \widehat{\phi}_i^0$ outside. If we would take derivative terms in $G(\phi_i)$ into account, this profile would be smoothened, and additional contributions to the action proportional to the length of the loop C would be obtained. As stated above, a treatment of these effects would require the solution of the combined equations of motion of ϕ_i, $F^n_{\mu\nu}$ and $B^n_{\mu\nu}$ (in some similarity to the much simpler scenario [6] based on the dual abelian Higgs model) and depend evidently on the unknown bare action S_{bare} in (3.33) or G_{bare} in (3.35), (3.37).

In order to close this section we briefly report on our fruitless search for bound states in the model in the confining phase. In principle these would show up as poles in the propagators of the fields ϕ_i for $q^2 < 0$. To this end one has to develop the fields ϕ_i around the vacuum configuration,

$$\phi_i(x) = \widehat{\phi}_i^0 + \delta\phi_i(x)\,. \tag{4.31}$$

Inserting (4.31) into the action (4.1) the propagators for ϕ_i are obtained from the terms quadratic in $\delta\phi_i$. It is sensible to study first the "mass matrix" M^2_{ij} in the space $\delta\phi_i$:

$$M^2_{ij} = \frac{N}{2} \left. \frac{\delta^2 G(\phi_i)}{\delta\phi_i\,\delta\phi_j} \right|_{\widehat{\phi}_i^0} . \tag{4.32}$$

We recall that $G(\phi_i)$ is composed out of G_{bare} and ΔG, cf. eq. (3.43), with ΔG given in (3.41) for $k^2 \to 0$, and the $\widehat{\phi}_i^0$ satisfy (3.49). Due to the non-analytic behaviour of ΔG in the confining phase one finds that all entries of M^2_{ij} actually diverge for $k^2 \to 0$, but M^2_{ij} has two finite eigenvalues (depending on G_{bare}) with eigenvectors given by

$$\begin{pmatrix} \sqrt{\widehat{\phi}_1^0} \\ \sqrt{\widehat{\phi}_3^0} \\ 0 \end{pmatrix} \delta\phi_a\,, \qquad \begin{pmatrix} 0 \\ \sqrt{\widehat{\phi}_1^0} \\ \sqrt{\widehat{\phi}_3^0} \end{pmatrix} \delta\phi_b\,. \tag{4.33}$$

We have computed the full momentum dependence of the contribution $(\Delta P^{-1}(q^2))_{ab}$ to the inverse propagators, which arises from the quantum contribution ΔG (3.39) after expanding ϕ_i to second order in $\delta\phi_i$ in the "directions" specified by (4.33), for $|q^2| \lesssim \Lambda^2$. We found that $(\Delta P^{-1}(q^2))_{ab}$ depend only weakly (logarithmically) on q^2, and that its determinant is negative definite. Together with the negativity of $G_{\rm bare}(\phi_i)$ (recall the remark below eq. (3.35)) this implies that the full inverse propagators do not vanish for $|q^2| \lesssim \Lambda^2$, thus no bound-state pole appears at least below the UV cutoff.

Such a pole would actually be a disaster for unitarity: the negativity of $G(\phi_i)$, notably for $|\phi_i| \to \infty$, is harmless only if the fields ϕ_i are pure "auxiliary fields", i.e. with algebraic equations of motion.

On the other hand one may be deceived by the absence of "glueballs", if the present model is interpreted as an effective low energy theory for SU(N_c) Yang-Mills theory. However, we recall that the present model would only describe the abelian subsector of SU(N_c) Yang-Mills in the MAG, and that we finally have to add the (massive) "non-abelian" gluons W^a_μ as well. Our theory induces flux-tube like confining attractive interactions among all fields which couple like $J^n_{F,\mu\nu}, J^n_{B,\mu\nu}$ to F and B, hence the W^a_μ-gluons will necessarily form bound states which will correspond to the desired glueballs. (This scenario is quite consistent with heavy glueballs, given $M_W \sim \mathcal{O}(1\,\mathrm{GeV})$ in the MAG [14]).

Further details on the possible relation between the present model and SU(N_c) Yang Mills will be discussed in the next section.

5. Discussion and outlook

In this final section we want to discuss some general features of the model, notably the relation between the large-N limit and SU(N_c) Yang Mills at large N_c, and the systematic inclusion of higher derivatives in the "bare" action.

First, we had emphasized the solvability of the A_μ-$B_{\mu\nu}$–model for $N \to \infty$ in the sense that $W(J)$ or $\Gamma(A,B)$ can be given explicitely for arbitrary $G_{\rm bare}(\phi_i)$ as in (3.42) or (4.1), and one is left with the coupled equations of motion for A^n_μ, $B^n_{\mu\nu}$ and ϕ_i. To this end we had to assume a certain N-dependence of the coefficients of the bare action $S_{\rm bare}(F,B)$, cf. eq. (2.16):

$$S_{\rm bare}(F,B) = N\widehat{S}_{\rm bare}\left(\frac{F}{\sqrt{N}}, \frac{B}{\sqrt{N}}\right) \tag{5.1}$$

where the coefficients of $\widehat{S}_{\rm bare}$ are N-independent.

Let us now assume that $S_{\rm bare}(F,B)$ has been obtained from SU(N_c) Yang-Mills after integrating out the "non-abelian" gluons W^a_μ as in eq. (2.9). From a simple analysis of the contributing Feynman diagrams and employing $\alpha_s \sim N_c^{-1}$ one obtains, in the large-N_c limit,

$$S^{\rm YM}_{\rm bare}(F,B) = N_c^2\, \widehat{S}^{\rm YM}_{\rm bare}\left(\frac{F}{\sqrt{N_c}}, \frac{B}{\sqrt{N_c}}\right) \tag{5.2}$$

in contrast to (5.1). In principle it is still possible to introduce the operators $\mathcal{O}_i$ (2.12), and to express $\exp(-S^{\rm YM}_{\rm bare}(\mathcal{O}_i))$ in terms of a path integral over ϕ_i as in (3.33):

$$e^{-S^{\rm YM}_{\rm bare}(\mathcal{O}_i)} = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi_i \, e^{-N_c^2 G_{\rm bare}(\phi_i/N_c) - \int d^4x \phi_i \mathcal{O}_i} \,. \tag{5.3}$$

The powers of N_c on the right-hand side of (5.3) have been introduced such that, in the stationary point approximation, the coefficients of $G_{\rm bare}$ are of $\mathcal{O}(1)$. (This becomes clear after a rescaling $\phi_i \to N_c \phi_i$, $\mathcal{O}_i \to N_c \mathcal{O}_i$ and using (5.2)). However, now the stationary point approximation of the path integral (5.3) can no longer be justified in the large-N_c limit.

In addition, the structure of the operators $\mathcal{O}_i$ (2.12) assumes a global O(N) symmetry which corresponds to a rotation in the space of the $N = N_c - 1$ U(1) subgroups of SU(N_c) (or among the fields A^n_μ, $B^n_{\mu\nu}$, $n = 1, \ldots, N$). This O(N) symmetry is *not* a subgroup of SU(N_c) and is generally not a symmetry of $S^{\rm YM}_{\rm bare}(F, B)$. (It appears as an accidental global symmetry, however, if one assumes that $S^{\rm YM}_{\rm bare}(F, B)$ contains just quadratic SU(N_c) invariants.) In general it is thus necessary to introduce N copies of each operator $\mathcal{O}_i$ in (2.12),

$$\mathcal{O}^{(n)}_{i=1} = F^n_{\mu\nu} F^n_{\mu\nu} \quad \text{etc. for } i = 2, 3 \tag{5.4}$$

without sum over n on the right-hand side (or N independent linear combinations of $\mathcal{O}^{(n)}_i$).[2] In particular for small N the introduction of operators $\mathcal{O}^{(n)}_i$ is straightforward, but again the stationary point approximation of the path integral operators $\mathcal{O}^{(n)}_i$ (5.3) can no longer be supported by a large-N_c limit.

We emphasize, however, that the essential phenomena described by the present class of models do not depend on the large-N limit (as the phenomenon of spontaneous symmetry breaking in bosonic O(N) models): these are the appearance of a confining phase, associated with an approximate (low energy) vector gauge symmetry and duality transformation, and associated with the area law for the Wilson loop. They have their origin in the convexity of a quantum effective action in spite of a non-convex bare action, which holds independently of a large-N limit. Just the precise relation between the effective action and the bare action can only be obtained for $N \to \infty$. An application of the present approach to SU(N_c) Yang Mills will thus require — apart from a determination of $S^{\rm YM}_{\rm bare}$ as in eq. (2.9) — a performance of the $\mathcal{D}\phi_i$ path integrals beyond the stationary point approximation.

In general we cannot expect that a "bare" action as obtained by eq. (2.9) depends only on operators $\mathcal{O}_i$ of the form (2.12), since it will involve the most general terms with higher derivatives. On the one hand we recall that the functional $G_{\rm bare}(\phi_i)$ introduced in (3.34), which specifies the models solved in section 3, may well contain

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[2] Note that a Z_2 symmetry (charge conjugation of all W^a_μ together with $A^n_\mu \to -A^n_\mu$) *is* a subgroup of SU(N_c), which allows to restrict ourselves to bilinear operators.

arbitrary derivatives acting on ϕ_i. These functionals $G_{\rm bare}(\phi_i)$ correspond to bare actions $S_{\rm bare}(\mathcal{O}_i)$ with arbitrary derivatives acting on $\mathcal{O}_i$. On the other hand, these do not correspond to the most general form of higher derivative terms: in general it will be necessary to consider bilinear operators of the form

$$\mathcal{O}_{i,\lambda_1\cdots\lambda_i} = F_{\mu\nu}\, f_{\mu\nu\rho\sigma,\lambda_1\cdots\lambda_i}\, F_{\rho\sigma} \tag{5.5}$$

(and with F replaced by B) where f depends on Kronecker symbols or ε-symbols in the Lorentz indices *and* derivatives $\overset{\leftrightarrow}{\partial}_\mu$ (with $F_1\overset{\leftrightarrow}{\partial}_\mu F_2 \equiv F_1\partial_\mu F_2 - (\partial_\mu F_1)F_2$).

Nevertheless it is possible to proceed as before: one can introduce auxiliary fields $\phi_{i,\lambda_1\cdots\lambda_i}$ for each of the operators in (5.5), and rewrite the most general bare action $S_{\rm bare}(\mathcal{O}_i)$ as

$$e^{-S_{\rm bare}(\mathcal{O}_i)} = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi_i\, e^{-G_{\rm bare}(\phi_i)-\int d^4x\phi_i\mathcal{O}_i} \tag{5.6}$$

(adding, if one wishes, appropriate factors of N as in (3.34)). Of course the appropriate contraction of Lorentz indices is understood in $\phi_i\mathcal{O}_i$.

Inserting (5.6) into (3.33) one obtains an expression for $W_k(J)$ analogous to (3.36), and the $\mathcal{D}A\mathcal{D}B$ path integrals are still gaussian. The essential technical complication is now the construction of the propagators P^{rs} of the A_μ, $B_{\mu\nu}$ — system which appear in (3.38), (3.39), and the computation of $\Delta G(\phi_i)$ from (3.39) via a generalization of (3.40) with higher powers of p^2 in the arguments of the logarithms.

However, one quickly realizes that the appearance of a confining phase, i.e. a solution of the stationary point equations for ϕ_i of the form of eqs. (3.48) and (3.49) depends only on the infrared behaviour of the propagators $P^{rs}(p^2)$ for $p^2 \to 0$. This, in turn, is unchanged by the inclusion of operators $\mathcal{O}_i$ in (5.6) involving higher derivatives.

The general features of the model will thus remain untouched by the inclusion of more general operators of the form (5.5), but its "fine-structure" will be affected: Lorentz scalar auxiliary fields ϕ_i associated to Lorentz scalar operators $\mathcal{O}_i$ involving higher derivatives will generally have non-vanishing vevs and affect the vacuum structure, and auxiliary fields ϕ_i with Lorentz indices will affect the "response" of the model to sources $J_{F,\mu\nu}$, $J_{B,\mu\nu}$.

Since the inclusion of a large number of additional auxiliary fields ϕ_i is evidently a rather ambitious program[3] the question arises whether — and to what extent — one could make use of the property of "universality" of local quantum field theories as the ones presented here which, by definition, corresponds to features which are practically independent of the UV cutoff Λ and the bare action. The phenomenon

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[3] The introduction of pseudoscalar operators of the form $\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ etc., without higher derivatives, should however be feasable. We would not expect vevs of the associated auxiliary fields, which would spontaneously break parity, but their two point functions would give us informations on the topological susceptibility.

of universality arises in the response of a quantum field theory to external sources which induce only "low energy" phenomena: the momenta q_i of the sources have to satisfy $q_i^2 \ll \Lambda^2$, and the change ΔE of the vacuum energy (in the case of, e.g., constant sources) has to be much smaller than Λ. (Otherwise one is sensitive to the non-universal regime of the effective potential.)

First, the Wilson loop corresponds to a source which satisfies none of these criteria: even after rewriting it as a source located on a surface S bounded by the loop (as in (4.16)) this source varies rapidly in $x_\perp$ perpendicular to S, and its expectation value depends thus on the non-universal terms involving high derivatives in the effective action. Fortunately this does not affect the area law but, as we have seen in section 4, the computation of the string tension. In addition, even after replacing an infinitely thin surface S by a layer of finite thickness, the field configurations inside this layer will generically depend on the non-universal regime of the effective action unless the "source" is unconventionally weak (or "diluted").

The response of the present class of models to sources which vary sufficiently slowly in x and which are sufficiently weak will, on the other hand, not depend on details of $G_{\rm bare}$ and/or higher derivative terms. Light quarks in QCD (with extended wave functions) could play the role of such sources, and their coupling to the present model will constitute an interesting task in the future.[4]

To conclude, we have investigated a class of four-dimensional gauge theories with finite UV cutoff, which exhibit confinement and allow nevertheless — in the large-N limit — for controllable computations notably in the infrared regime. Some features of the confining phase correspond quite to our expectations, notably the possiblity to perform a duality transformation of the low energy part of the effective action and thus to interpret confinement as monopole condensation. A technically related but nevertheless unexpected feature is the appearance of a low energy vector gauge symmetry, which allows to "gauge away" the low momentum modes of the abelian gluons A^n_μ and explains the absence of the corresponding asymptotic states. If we assume in addition, as in eq. (2.15), higher derivative terms bilinear in $B^n_{\mu\nu}$ such that the corresponding massive poles in the propagators are absent, the model has no asymptotic states at all (even no bound states, cf. the end of section 5).

The only "meaning" of the model is thus its reaction to external sources or currents. In the case of conserved currents (with respect to the approximate vector gauge symmetry) confinement is then actually not based on a long-range attractive force between two distinct sources — the corresponding components of the propagators cancel precisely for conserved external currents as explained in the paragraphs below eq. (4.20) — but on a different phenomenon: the minimal geometric object which can be associated to a conserved current or source $J_{\mu\nu}$ is a surface S (cf. the

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[4] At least a computation of $\Gamma_{\rm eff}(\phi_i, A, B)$ in a systematic expansion in (∂_μ/M_W), beyond the large-N limit, should be feasable.

expressions (4.16) and (4.19)) and, as discussed in section 4, the contribution to the action of a source which is constant on S is proportional to S (at least for large S) inducing the area law of the Wilson loop. The tensor structure $T_{2,\mu\nu,\rho\sigma}$ in the propagator P^{BB} (cf. eqs. (A.2), (A.7)) is essential for this result. The question why a pointlike particle like a heavy quark can be associated to source for $B_{\mu\nu}$ which is constant on a surface S is returned to the underlying microscopic theory, such as the non-abelian sector of SU(N_c) Yang-Mills, but here we found that this is obligatory in order to be able to satisfy the constraints (4.22)

Interestingly, if the lattice results [10] on the masses of the non-abelian gluons in the MAG are correct, an elaboration of the precise relation between the present "effective low energy theory" and SU(N_c) Yang Mills — i.e. the structure of $S^{\text{YM}}_{\text{bare}}$ as well as the structure of the sources — would involve only physics at comparatively small scales of $\mathcal{O}((1\ \text{GeV})^{-1})$ and is thus perhaps feasable.

Apart from this evident application of our class of models we would like to point out that generalizations in the sense of [8, 24] suggest itself: it is fairly straightforward to vary (notably increase) the rank of the tensors A_μ, $B_{\mu\nu}$ (maintaining rank$(B) =$ rank$(A)+1$) and to vary the dimension of the space-time. This way one can develop dynamical models for the condensation of topological defects of various dimensions (p-branes) in various space-time dimensions.

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A. Explicit propagators

In this appendix we give explicit expressions for the different propagators, for constant fields ϕ_i and in the Landau gauge, both in momentum space and space-time and for various limits. For the cutoffs terms ΔS_k (cf. eq. (2.10)) we use the simplest form

$$\Delta S_k(A,B) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\left\{F^n_{\mu\nu}(p)R^A(p,k,\Lambda)F^n_{\mu\nu}(-p) + B^n_{\mu\nu}(p)R^B(p,k,\Lambda)B^n_{\mu\nu}(-p)\right\}. \tag{A.1}$$

(This form of the cutoff on the fields A^n_μ is consistent only in the Landau gauge.) Subsequently it will be convenient to define the following three 4-index tensors:

$$\begin{aligned} T_{1,\mu\nu,\rho\sigma}(p) &= \delta_{\mu\rho}p_\nu p_\sigma - \delta_{\mu\sigma}p_\nu p_\rho - \delta_{\nu\rho}\, p_\mu p_\sigma + \delta_{\nu\sigma}\, p_\mu p_\rho\,, \\ T_{2,\mu\nu,\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\,, \\ T_{3,\mu\nu,\rho\sigma}(p) &= T_{2,\mu\nu,\rho\sigma} - \frac{1}{p^2}T_{1,\mu\nu,\rho\sigma}(p)\,. \end{aligned} \tag{A.2}$$

First we give the three propagators P^{AA}, P^{AB} and P^{BB} in momentum space:

$$\begin{aligned}
P^{AA}_{\mu,\nu}(p) &= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) P^A(p^2)\,,\\
P^A(p^2) &= \frac{\sigma p^2 + 4(\phi_3 + R^B(p^2))}{4p^2 d_1(p^2)}\,,\\
d_1(p^2) &= \sigma p^2 \left(\phi_1 + R^A(p^2)\right) + 4\left(\phi_1 + R^A(p^2)\right)\left(\phi_3 + R^B(p^2)\right) - \phi_2^2\,,\\
P^{AB}_{\mu,\rho\sigma}(p) &= i\left(p_\rho \delta_{\mu\sigma} - p_\sigma \delta_{\mu\rho}\right) P^{AB}(p^2)\,,\\
P^{AB}(p^2) &= \frac{\phi_2}{2p^2 d_1(p^2)}\,,\\
P^{BB}_{\mu\nu,\rho\sigma}(p) &= T_{1,\mu\nu,\rho\sigma} \cdot P^{BB}(p^2) + T_{3,\mu\nu,\rho\sigma} \cdot \frac{1}{d_2(p^2)}\,,\\
P^{BB}(p^2) &= \frac{\phi_1 + R^A(p^2)}{4p^2 d_1(p^2)}\,,\\
d_2(p^2) &= hp^2 + 4R^B(p^2) + 4\phi_3\,.
\end{aligned} \tag{A.3}$$

For later use it is convenient to introduce the propagators P^{FF}, P^{FB} defined as

$$P^{FF}_{\mu\nu,\rho\sigma}(p) = p_{[\mu} P^{AA}_{\nu],[\rho}\, p_{\sigma]} = T_{1,\mu\nu,\rho\sigma}(p)\ P^A(p^2)\,, \tag{A.4a}$$

$$P^{FB}_{\mu\nu,\rho\sigma}(p) = p_{[\mu,} P^{AB}_{\nu],\rho\sigma} = T_{1,\mu\nu,\rho\sigma}(p)\ P^{AB}(p^2)\,. \tag{A.4b}$$

Many physical phenomena induced by the effective action of our model are best understood in terms of the propagators in ordinary (euclidean) space-time in the limit where the infrared cutoff is removed. For simplicity we give these propagators also for vanishing UV cutoff (which is anyhow non-universal); their short-distance singularities should be regularized correspondingly.

The large distance behaviour of the space-time propagators depends often crucially on the combination $\Sigma = (4\phi_1\phi_3 - \phi_2^2)/\sigma\phi_1$ (defined also in eq. (3.43)), which vanishes in the "confining phase" (cf. section 3). For completeness we give the space-time propagators both for finite Σ and for $\Sigma \to 0$.

First, for finite Σ, we have

$$\begin{aligned}
P^{AA}_{\mu,\nu}(z) = \frac{1}{16\pi^2\sigma\phi_1^2} &\left\{ \delta_{\mu\nu} \left(\frac{4\phi_1\phi_3}{\Sigma|z|^2} - \frac{\phi_2^2}{\sqrt{\Sigma}|z|} K_1(|z|\sqrt{\Sigma}) \right) - \right.\\
&\left. -\partial_\mu \partial_\nu \left(\frac{\phi_2^2}{\Sigma^2|z|^2} + \frac{2\phi_1\phi_3}{\Sigma} \ell n|z| - \frac{\phi_2^2}{\Sigma^{3/2}|z|} K_1(|z|\sqrt{\Sigma}) \right) \right\},\\
P^{AB}_{\mu,\rho\sigma}(z) = \frac{\phi_2}{8\pi^2\sigma\phi_1} &\left(\partial_\rho \delta_{\mu\sigma} - \partial_\sigma \delta_{\mu\rho}\right) \left\{ \frac{1}{\Sigma|z|^2} - \frac{1}{\sqrt{\Sigma}|z|} K_1(|z|\sqrt{\Sigma}) \right\},\\
P^{BB}_{\mu\nu,\rho\sigma}(z) = \frac{-1}{4\pi^2\sigma} &T_{1,\mu\nu,\rho\sigma}(\partial_z) \left\{ \frac{1}{\Sigma|z|^2} - \frac{1}{\sqrt{\Sigma}|z|} K_1(|z|\sqrt{\Sigma}) \right\} +\\
&+ \frac{1}{16\pi^2\phi_3} T_{1,\mu\nu,\rho\sigma}(\partial_z) \left\{ \frac{1}{|z|^2} - \frac{2}{|z|} \sqrt{\frac{\phi_3}{h}} K_1\left(|z| 2 \sqrt{\frac{\phi_3}{h}} \right) \right\} +
\end{aligned}$$

$$+\frac{\sqrt{\phi_3}}{2\pi^2 h^{3/2}}T_{2,\mu\nu,\rho\sigma}\frac{1}{|z|}K_1\left(|z|\sqrt{\frac{\phi_3}{h}}\right). \tag{A.5}$$

$T_{1,\mu\nu,\rho\sigma}(\partial_z)$ is defined in (A.2) with the replacement $p_\mu \to \partial_\mu$ and K_1 is a Bessel function. The propagators P^{FF} and P^{FB} are

$$P^{FF}_{\mu\nu,\rho\sigma}(z) = \frac{-1}{16\pi^2\sigma\phi_1^2}T_{1,\mu\nu,\rho\sigma}(\partial_z)\left\{\frac{4\phi_1\phi_3}{\Sigma|z|^2} - \frac{\phi_2^2}{\sqrt{\Sigma}|z|}K_1(|z|\sqrt{\Sigma})\right\}, \tag{A.6a}$$

$$P^{FB}_{\mu\nu,\rho\sigma}(z) = \frac{\phi_2}{8\pi^2\sigma\phi_1}T_{1,\mu\nu,\rho\sigma}(\partial_z)\left\{\frac{1}{\Sigma|z|^2} - \frac{1}{\sqrt{\Sigma}|z|}K_1(|z|\sqrt{\Sigma})\right\}. \tag{A.6b}$$

Finally, in the limit $\Sigma \to 0$ the space-time propagators become (we also replace ϕ_2 by $2\sqrt{\phi_1\phi_3}$)

$$\begin{aligned}
P^{AA}_{\mu,\nu}(z) &= \frac{1}{16\pi^2\phi_1}\left\{\delta_{\mu\nu}\left(\frac{1}{|z|^2} - \frac{2\phi_3}{\sigma}\log|z| + \text{const.}\right) - \right.\\
&\qquad \left. -\frac{1}{2}\partial_\mu\partial_\nu\left(\log|z| - \frac{\phi_3|z|^2}{2\sigma}(\log|z| + \text{const.'})\right)\right\},\\
P^{AB}_{\mu,\rho\sigma}(z) &= \frac{-\sqrt{\phi_3}}{8\pi^2\sigma\sqrt{\phi_1}}\left(\partial_\rho\delta_{\mu\sigma} - \partial_\sigma\delta_{\mu\rho}\right)\log|z|,\\
P^{BB}_{\mu\nu,\rho\sigma}(z) &= \frac{1}{8\pi^2\sigma}T_{1,\mu\nu,\rho\sigma}(\partial_z)\log|z|\\
&\quad + \frac{1}{16\pi^2\phi_3}T_{1,\mu\nu,\rho\sigma}(\partial_z)\left\{\frac{1}{|z|^2} - \frac{2}{|z|}\sqrt{\frac{\phi_3}{h}}K_1\left(|z|2\sqrt{\frac{\phi_3}{h}}\right)\right\}\\
&\quad + \frac{\sqrt{\phi_3}}{2\pi^2 h^{3/2}}T_{2,\mu\nu,\rho\sigma}\frac{1}{|z|}K_1\left(|z|2\sqrt{\frac{\phi_3}{h}}\right).
\end{aligned} \tag{A.7}$$

The expressions const. and const.′ in (A.7) actually diverge like $\log(\Sigma)$. A finite result is obtained, however, for P^{FF} and P^{FB} in the limit $\Sigma \to 0$:

$$P^{FF}_{\mu\nu,\rho\sigma}(z) = \frac{-1}{16\pi^2\phi_1}T_{1,\mu\nu,\rho\sigma}(\partial_z)\left\{\frac{1}{|z|^2} - \frac{2\phi_3}{\sigma}\log|z|\right\}, \tag{A.8a}$$

$$P^{FB}_{\mu\nu,\rho\sigma}(z) = \frac{-1}{8\pi^2\sigma}\sqrt{\frac{\phi_3}{\phi_1}}T_{1,\mu\nu,\rho\sigma}(\partial_z)\log|z|. \tag{A.8b}$$

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